Single-Slit Diffraction—C.E. Mungan, Spring 2012

This document summarizes approximate and exact values of three parameters characterizing the secondary maxima of a single-slit pattern (in the Fraunhofer limit): the peak angular positions, heights (in intensity), and areas. Label the orders by $m = 1,2,3,\ldots$.

Start with the approximate model. For small angles (in rad), the peak positions are approximately

$$\theta_m = \left(m + \frac{1}{2} \right) \frac{\lambda}{a} \tag{1}$$

for incident monochromatic light of wavelength λ passing through a slit of width a. Considering the first-order maximum, one can chop the slit into thirds and state that the 1st third interferes destructively with the 2nd third, leaving only the electric field from the 3rd third to reach the screen. Since intensity is proportional to the field squared, one might suppose the relative intensity of the first maximum to be one-ninth of the central maximum, but that is wrong for the following reason. To get destructive interference between adjacent thirds of the slit, there must be a half-cycle phase difference between rays passing through the slit that are separated laterally by a distance of one-third the slit width. But now consider a ray 1 that just skirts the upper edge of the surviving 3rd third of the slit and a ray 2 that just skirts its lower edge. Those two rays are separated laterally by one-third the slit width and thus must be half a cycle different in optical phase. Therefore the average relative electric field passing through the slit in first order must be one-third of the average value over a half-cycle of a sinusoidal wave,

$$\frac{E_1}{E_0} = \frac{1}{3} \cdot \frac{1}{\pi} \int_0^{\pi} \sin\alpha \, d\alpha = \frac{1}{3} \cdot \frac{2}{\pi} \,. \tag{2}$$

Squaring this result and generalizing it to any secondary maximum, we get

$$\frac{I_m}{I_0} = \left[\frac{2/\pi}{2m+1} \right]^2 . \tag{3}$$

Finally, we note that the angular distance between adjacent minima of positive order is half that between the $m = -1$ and $m = +1$ minima that bracket the central maximum. Thus the relative area of the secondary peaks is estimated to be

$$\frac{A_m}{A_0} = \frac{I_m \Delta\theta_m}{I_0 \Delta\theta_0} = \frac{1}{2} \left[\frac{2/\pi}{2m+1} \right]^2 . \tag{4}$$

Numerical values of Eqs. (1), (3), and (4) for the first four secondary maxima are listed in the table below.

order m	angular position (units of λ/a)	peak intensity (units of I_0)	area (units of A_0)
1	1.5	1/22.2 = 4.50%	1/44.4
2	2.5	1/61.7 = 1.62%	1/123
3	3.5	1/121 = 0.83%	1/242
4	4.5	1/200 = 0.50%	1/400

We compare these to the exact values. The angular positions are

$$\theta_m = \sin^{-1}\left(\frac{\lambda}{\pi a}\alpha_m\right) \approx \frac{\lambda}{a} \cdot \frac{\alpha_m}{\pi} \tag{5}$$

assuming they are small (i.e., $\lambda \ll a$). Here α_m are the solutions of the equation $\tan(\alpha_m) = \alpha_m$, obtained by typing into WolframAlpha the command

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FindRoot[Tan[x]==x,{x,N}]
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where N is a starting guess somewhat smaller in value than the approximate value $(2m+1)\pi / 2$. The relative intensities are then

$$\frac{I_m}{I_0} = \text{sinc}^2\,\alpha_m \tag{6}$$

with areas of

$$\frac{A_m}{A_0} = \frac{\int_{m\pi}^{(m+1)\pi} \text{sinc}^2\,\alpha\, d\alpha}{\int_{-\pi}^{\pi} \text{sinc}^2\,\alpha\, d\alpha} \tag{7}$$

evaluated in WolframAlpha. Numerical values of Eqs. (5) to (7) are listed in the next table. We see that the approximate values in the previous table are reasonably good estimates.

order m	angular position (units of λ/a)	peak intensity (units of I_0)	area (units of A_0)
1	1.430	1/21.2 = 4.72%	1/38.3
2	2.459	1/60.7 = 1.65%	1/110
3	3.471	1/120 = 0.83%	1/217
4	4.477	1/199 = 0.50%	1/359