Evaluating a Common Integral—C.E. Mungan, Fall 2004

In the solution of the soap-bubble problem (Taylor P6.19), one gets the following second integral after applying the Euler-Lagrange equation,

$$x = c \int \frac{dy}{\sqrt{y^2 - c^2}}. \tag{1}$$

Whenever one sees integrals involving square roots of squares of the integration variable, a good thing to try is a trignometric substitution, $y/c = \text{trigA}(\theta)$. Choose this trig function so that the square root reduces to some other single trig function, $\text{trigB}(\theta)$. In the present case, this means we want

$$\text{trigA}^2(\theta) - 1 = \text{trigB}^2(\theta). \tag{2}$$

This is satisfied for the choice $\text{trigA}(\theta) = \sec\theta$ and $\text{trigB}(\theta) = \tan\theta$, as you can see by dividing the following identity through by $\cos^2\theta$,

$$1 - \cos^2\theta = \sin^2\theta. \tag{3}$$

So let's substitute $y = c\sec\theta$ into Eq. (1) to get

$$x = c \int \frac{d(c\sec\theta)}{c\tan\theta} = c \int \frac{\frac{d}{d\theta}\sec\theta}{\tan\theta} d\theta. \tag{4}$$

But

$$\frac{d}{d\theta}\sec\theta = \frac{d}{d\theta}\frac{1}{\cos\theta} = \left(\frac{-1}{\cos^2\theta}\right)(-\sin\theta) = \tan\theta\sec\theta \tag{5}$$

so that Eq. (4) simplifies to

$$x = c \int \sec\theta\, d\theta = c\ln(\sec\theta + \tan\theta) + d. \tag{6}$$

Please check that the derivative of the righthand side does in fact give you the integrand. Now in order to return to the original variable y, rewrite Eq. (6) as

$$x = c\ln\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right) + d = c\ln\left(\frac{y}{c} + \sqrt{\frac{y^2}{c^2} - 1}\right) + d \tag{7}$$

and therefore

$$\exp\left(\frac{x - d}{c}\right) - \frac{y}{c} = \sqrt{\frac{y^2}{c^2} - 1}. \tag{8}$$

Square both sides and simplify to get

$$\exp\left(2\frac{x - d}{c}\right) + 1 = 2\frac{y}{c}\exp\left(\frac{x - d}{c}\right). \tag{9}$$

Divide both sides by $2\exp(x/c - d/c)$ to get the final answer,

$$\frac{y}{c} = \cosh\left(\frac{x-d}{c}\right) \tag{10}$$

which is a curve known as a "catenary." It also describes the shape of an unstretched chain hanging between two fixed points.

Whew, that was a lot of algebra to get the answer. Is there an easier way? The answer is yes. In hindsight, we should have done a hyperbolic substitution instead of a trignometric substitution, namely try $y = c\cosh u$ in Eq. (1),

$$x = c\int \frac{d\cosh u}{\sqrt{\cosh^2 u - 1}} = c\int \frac{\sinh u}{\sinh u}\,du = c\,u + d. \tag{11}$$

Wow, how much easier! Now all we have to do is return to the variable y,

$$\cosh\frac{x-d}{c} = \cosh u = \frac{y}{c} \tag{12}$$

which is Eq. (10)! The values of the two integration constants, c and d, are found by fitting to the two boundary values (x_1, y_1) and (x_2, y_2) through which the soap film must pass. This turns out to be a bit messy, requiring that we graphically solve a transcendental equation, so we won't pursue that here.

So why do the first method at all, if it's so much work? Well, it demonstrates that you can do integrals different ways and get the same solution, although some ways are of course easier than others. Equally importantly, it means that a calculator or computer has to make a choice about which form to present for its answer. Ignoring the indefinite constant of integration d and choosing $c = 1$ for simplicity, note that Eqs. (7) and (11) imply that

$$\int \frac{dy}{\sqrt{y^2 - 1}} = \ln\left(y + \sqrt{y^2 - 1}\right) \tag{13}$$
$$= \cosh^{-1}(y)$$

I bet your calculator chooses the first form. But if we were doing it by hand, we would use the second method and thereby save ourselves a lot of subsequent algebra to get the solution into the form of Eq. (10).