

### Solution of a Cubic Equation—C.E. Mungan, Summer 2013

*Ref: J. Derbyshire, Unknown Quantity (Joseph Henry Press, 2006)*

We wish to solve the cubic

$$ax^3 + bx^2 + cx + d = 0 . \quad (1)$$

Step 1: The coefficient  $a$  cannot be zero, because otherwise this equation would not be a cubic.

So divide through by it to get

$$x^3 + Bx^2 + Cx + D = 0 . \quad (2)$$

Step 2: Complete the cube by defining  $X \equiv x + B/3$  to get the depressed cubic

$$X^3 + pX + q = 0 \quad (3)$$

where

$$p \equiv C - B^2/3 \quad (4)$$

and

$$q \equiv D - BC/3 + 2B^3/27 . \quad (5)$$

Step 3: Introduce two new variables such that  $X \equiv u + v$ . Substituting into Eq. (3) gives

$$u^3 + v^3 + 3uv(u + v) = -q - p(u + v) . \quad (6)$$

Since there are two new unknowns, we can choose two independent equations for them. By inspection of Eq. (6), choose to separately equate the first part and the second part of each side of that equation to get

$$u^3 + v^3 = -q \quad (7)$$

and

$$3uv(u + v) = -p(u + v) . \quad (8)$$

Solve Eq. (8) for  $v$  to obtain

$$v = -p/3u \quad (9)$$

and substitute that into Eq. (7) to find

$$u^6 + qu^3 - p^3/27 = 0 . \quad (10)$$

This is a quadratic equation in  $u^3$  and so it has two solutions. Noting that  $u$  and  $v$  appear symmetrically in Eqs. (7) and (8), these two solutions must be the primary roots for  $u$  and  $v$ , so that

$$u = \left[ \frac{-q + \sqrt{q^2 + 4p^3/27}}{2} \right]^{1/3} \quad (11)$$

and

$$v = \left[ \frac{-q - \sqrt{q^2 + 4p^3/27}}{2} \right]^{1/3} . \quad (12)$$

Step 4: Denote the three cube roots of unity as 1,  $\omega$ , and  $\omega^2$  where

$$\omega \equiv e^{i2\pi/3} = \frac{-1+i\sqrt{3}}{2} \quad (13)$$

and

$$\omega^2 = e^{i4\pi/3} = \frac{-1-i\sqrt{3}}{2}. \quad (14)$$

Then the three solutions of Eq. (3) are

$$X_1 = u + v, \quad (15a)$$

$$X_2 = \omega u + \omega^2 v, \text{ and} \quad (15b)$$

$$X_3 = \omega^2 u + \omega v. \quad (15c)$$

These have been chosen so that the coefficient of  $u$  multiplied by the coefficient of  $v$  equals 1, in order that  $uv = -p/3$  in accordance with Eq. (9).