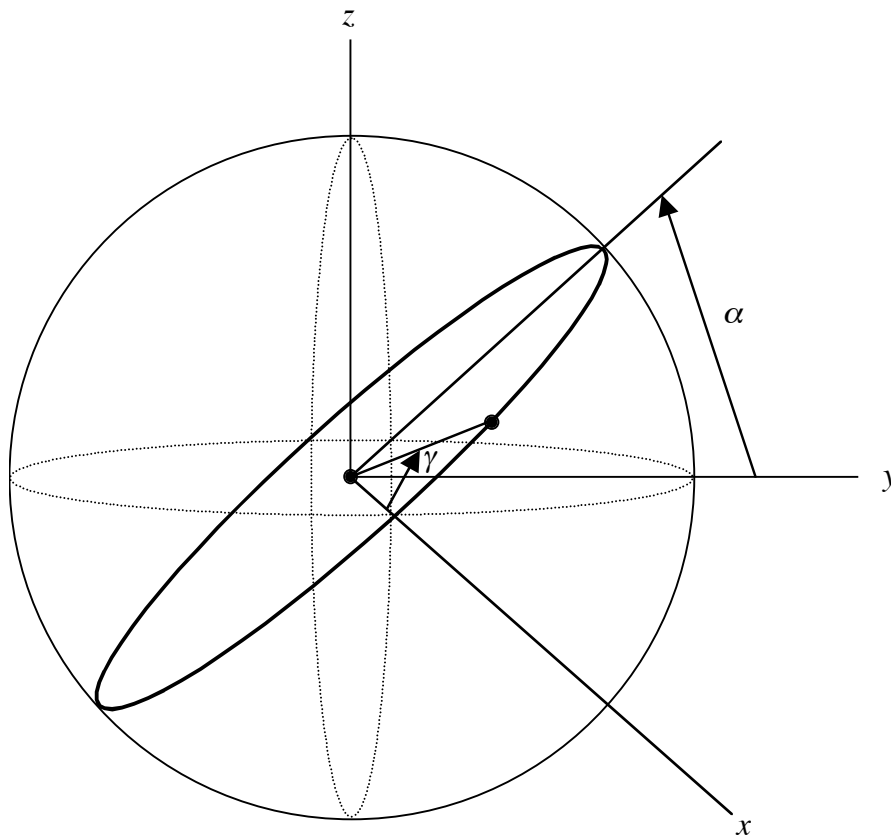


Equilateral Triangle on the Surface of a Sphere—C.E. Mungan, Fall 2004

The purpose of this exercise is to compute the interior angle α and area A of an equilateral triangle on the surface of a sphere of unit radius. That is, the triangle has 3 sides of *given* equal length s , each of which is a portion of a great circle.

Without loss of generality, choose the base of the triangle to lie along the equator (latitude¹ $\theta = 0$), let the western-most vertex lie on the Greenwich meridian (longitude $\phi = 0$), and suppose that the triangle is in the northern hemisphere. We then get the following diagram.



The horizontal circle is the equator (on which the bottom side of the triangle is located), the vertical circle is the Greenwich meridian, and the western side of the triangle is located along the bold diagonal circle, which is tilted at angle α to the horizontal. As one can see from the diagram, this is the same as the angle between the bold circle and the equator that we are trying to find. (To keep the diagram uncluttered, I have not drawn in the eastern side of the triangle.)

One of the two dots on the diagram denotes the center of the sphere and the other indicates an arbitrary point on the western edge of the triangle. The coordinates of this latter point can be expressed in 3 different systems: longitude and latitude (θ, ϕ) ; Cartesian (x, y, z) where the x axis

intersects the Greenwich meridian on the equator and the z axis points through the north pole; and finally, the rotation angle γ around the bold great circle. The mathematical relation between these 3 sets of coordinates is

$$x = \cos \gamma = \cos \theta \cos \phi, \quad (1)$$

$$y = \sin \gamma \cos \alpha = \cos \theta \sin \phi, \text{ and} \quad (2)$$

$$z = \sin \gamma \sin \alpha = \sin \theta. \quad (3)$$

As a consistency check, it is easy to verify that the second half of Eq. (3), giving a relation between the various angles, is consistent with the second half of Eqs. (1) and (2).

Dividing Eq. (2) by (1) gives

$$\tan \gamma \cos \alpha = \tan \phi. \quad (4)$$

But we reach the northern vertex of the triangle when $\gamma = s$ and $\phi = s/2$, since the sphere has unit radius. Substituting these limits into Eq. (4) gives an expression for the interior angle,

$$\boxed{\cos \alpha = \frac{\tan \frac{s}{2}}{\tan s}}. \quad (5)$$

Consider three interesting special cases of this result:

(i) As $s \rightarrow 0$ then

$$\cos \alpha \cong \frac{s/2}{s} \Rightarrow \alpha = 60^\circ \quad (6)$$

as expected for a flat equilateral triangle.

(ii) When $s = \pi/2$ then $\cos \alpha = 1/\infty \Rightarrow \alpha = 90^\circ$, corresponding to one octant of the sphere.

(iii) Finally when $s = 2\pi/3$ then $\cos \alpha = -1 \Rightarrow \alpha = 180^\circ$. In this case the three legs of the “triangle” span the equator, giving the largest possible surface area, namely half of the sphere.

Next we compute the area of the triangle. If we bisect the triangle east-west, then one can see that the bounding curves of the western half-triangle are $\theta = 0$ along the bottom, $\phi = s/2$ along the eastern edge, and the bold circle along the western edge. By trial and error, it proves easier to do the theta integration first,¹

$$A = 2 \int_0^{s/2} \int_0^{\theta(\phi)} \cos \theta d\theta d\phi = 2 \int_0^{s/2} \sin \theta(\phi) d\phi. \quad (7)$$

To find the curve $\theta(\phi)$ describing the bold circle, divide Eq. (3) by (2) to get

$$\tan \alpha \sin \phi = \tan \theta = \frac{1}{\sqrt{\sin^{-2} \theta(\phi) - 1}}. \quad (8)$$

Substituting this into Eq. (7) gives

$$A = 2 \int_0^{s/2} \frac{d\phi}{\sqrt{1 + \cot^2 \alpha \csc^2 \phi}}. \quad (9)$$

This can be integrated by making the substitution² $\sin \psi \equiv \sin \alpha \cos \phi$ to get

$$A = 2 \left[\alpha - \sin^{-1}(\sin \alpha \cos \frac{s}{2}) \right]. \quad (10)$$

Using Eq. (5) and the double-angle formulae, one can show that

$$\cos \frac{s}{2} = \frac{\cos \frac{\alpha}{2}}{\sin \alpha} \Rightarrow \sin^{-1}(\sin \alpha \cos \frac{s}{2}) = \frac{\pi}{2} - \frac{\alpha}{2} \quad (11)$$

so that one gets the wonderfully compact answer³

$$\boxed{A = 3\alpha - \pi}. \quad (12)$$

Consider what this gives for the above special cases:

(i) To the lowest nonzero order as $s \rightarrow 0$, Eq. (6) implies that $\alpha \rightarrow 60^\circ + \delta$ where δ is small, and thus Eq. (5) implies

$$\cos 60^\circ - (\sin 60^\circ) \delta \cong \frac{(\frac{s}{2} - \frac{s^3}{48}) / (1 - \frac{s^2}{8})}{(s - \frac{s^3}{6}) / (1 - \frac{s^2}{2})} \Rightarrow -\sqrt{3} \delta \cong -\frac{s^2}{24} + \frac{s^2}{8} + \frac{s^2}{6} - \frac{s^2}{2}. \quad (13)$$

Inserting this into Eq. (12) gives

$$A \rightarrow \frac{\sqrt{3}}{4} s^2 \quad (14)$$

which is the correct answer for a flat equilateral triangle of side s .

(ii) When $\alpha = 90^\circ$ then $A = \pi / 2$, which is indeed the area of one octant of a unit sphere.

(iii) Finally $\alpha = 180^\circ$ implies $A = 2\pi$, the surface area of a hemisphere.

1. The latitude is 90° minus the polar angle (often called the colatitude and, at the risk of considerable confusion, also usually designated by the spherical angle θ). Simply interchange $\cos \theta$ and $\sin \theta$ to switch between latitude and colatitude in this paper.

2. My thanks to David Bowman for suggesting this substitution.

3. You will probably not be surprised to learn that such a simple final result can be obtained much more easily. In fact, no calculus is required at all. See for example the discussion of spherical triangles at <http://math.rice.edu/~pcmi/sphere/gos4.html>.