Transition from Spinning to Rolling—C.E. Mungan, Fall 2022

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A rough solid sphere is lifted into the air, given top spin, and then placed on a horizontal surface. After it starts rolling without slipping, what percentage of the initial mechanical energy was dissipated away thermally?

Interestingly the answer does not depend on the mass *M*, radius *R*, or coefficient of kinetic friction μ of the sphere (provided the latter is nonzero). Assuming rolling friction and air drag are negligible, once the sphere starts rolling without slipping there is no further dissipation of mechanical energy. Static friction dissipates no mechanical energy, and the static frictional force is zero in any case for an object rolling without friction on a horizontal surface. (It cannot point forward because that would increase the translational velocity. Nor can it point backward because that would increase the angular velocity. So it must be zero!) Thus only kinetic friction is relevant to the problem and only during the time interval *t* between when the sphere is placed on the surface and when it starts to roll without slipping.

Specifically, during that time interval, the kinetic frictional force $f = \mu Mg$ must point forward, so that it increases the translational speed of the center of mass (CM) from $\upsilon_\text{i} = 0$ to its final value υ_f while simultaneously decreasing the angular speed about the CM from ω_i to ω_f. At the end of this time interval, the sphere will begin to roll without slipping so that $\upsilon_\text{f} = \omega_\text{f} R$.

Since the linear and angular accelerations are both constant during the time *t* when kinetic friction is operative, we can use kinematics. Newton's second law (N2L) for translations implies that the constant linear acceleration of the CM is $a = f / M = \mu g$ so that

$$\upsilon_\text{f} = \upsilon_\text{i} + at \quad \Rightarrow \quad t = \frac{\omega_\text{f} R}{\mu g}. \tag{1}$$

Likewise N2L for rotations says the constant angular acceleration about the CM is given by the ratio of the frictional torque and the moment of inertia $I = \gamma MR^2$ where γ is the numerical shape factor of the rolling object (such as 2/5 for a solid sphere),

$$\alpha = \frac{-fR}{\gamma MR^2} \tag{2}$$

where the minus sign assumes that the top spin is chosen to define the positive angular direction. Therefore

$$\omega_\text{f} = \omega_\text{i} + \alpha t = \omega_\text{i} - \frac{\omega_\text{f}}{\gamma} \quad \Rightarrow \quad \boxed{\omega_\text{f} = \frac{\gamma}{1+\gamma}\omega_\text{i}} \tag{3}$$

using Eqs. (1) and (2) along with $f = \mu Mg$. Here we see the remarkable cancellation of all dependence on *M*, *R*, and μ. In the case of a solid sphere, Eq. (3) tells us the final angular speed is a mere two-sevenths of the initial spin imparted to it, so that 71% of the angular speed was lost.

The initial and final mechanical energies are the total kinetic energies

$$K_\text{i} = \frac{1}{2} I \omega_\text{i}^2 = \frac{\gamma}{2} MR^2 \omega_\text{i}^2 \tag{4}$$

and

$$K_{\text{f}} = \frac{1}{2}I\omega_{\text{f}}^2 + \frac{1}{2}M\upsilon_{\text{f}}^2 = \boxed{\frac{\gamma}{1+\gamma}K_{\text{i}}} \quad (5)$$

respectively, after simplifying. Thus a shocking 71% of the mechanical energy is lost for a solid sphere! More generally, the fraction lost is $(1+\gamma)^{-1}$ and so almost all of the kinetic energy would be dissipated away if we embedded a heavy metal core at the center of a low-density cylinder or sphere.