Prove that $\sqrt{p/q}$ (where $p$ and $q$ are relatively prime*) is irrational if $p$ or $q$ is not a perfect square.

*Definitions: counting numbers are positive integers $\{1, 2, 3, \ldots\}$; some counting number $y$ is said to divide or be a factor of some counting number $x$ if $x/y = z$ where $z$ is a counting number; common factors of two terms are counting numbers greater than 1 which divide both terms; two numbers are said to be relatively prime if they are counting numbers having no common factors. For example, 4 and 35 are relatively prime, as are 1 and 10, while 4 and 26 are not.

Proof by contradiction: Assume that $\sqrt{p/q}$ is rational, where $p$ and $q$ are relatively prime and at most one of them is a perfect square. Then we can write

$$\frac{p}{q} = \frac{x}{y} \quad \text{where} \ x \text{ and } y \text{ are relatively prime}.$$  \hfill (1)

But this last clause requires that $x^2$ and $y^2$ must also be relatively prime, because any prime factor of $x^2$ is also a prime factor of $x$ and vice-versa, and likewise for $y^2$ and $y$. (Careful! A composite factor of $x^2$ need not be a factor of $x$. For example, 50 is a factor of 100 but not of 10, but 5 and 2 are factors of both.) We will find a contradiction to this requirement.

To do so, we start by rearranging Eq. (1) to get

$$py^2 = qx^2. \quad \text{ (2)}$$

Thus, $q$ divides $py^2$. But $q$ does not divide $p$ by assumption. Therefore, $q$ divides $y^2$ and we can write

$$y^2 = nq \quad \text{where} \ n \text{ is some counting number}. \quad \text{ (3)}$$

By the same argument, we can also conclude that

$$x^2 = mp. \quad \text{ (4)}$$

Substituting Eqs. (3) and (4) into (2) and simplifying shows that

$$m = n \equiv N. \quad \text{ (5)}$$

But $N$ cannot be equal to 1, because if it were, then Eqs. (3) and (4) would imply that $p = x^2$ and $q = y^2$ so that both are perfect squares, which we assumed not to be the case. We now have a contradiction because we see from Eqs. (3) and (4) that $N$ is a common factor of $x^2$ and $y^2$.

Comment: This proves for example that $\sqrt{3}$ is irrational!