Irrationality of Square Roots—C.E. Mungan, Fall 1999

Prove that $\sqrt{p/q}$ (where p and q are relatively prime*) is irrational if p or q is not a perfect square.

*Definitions: counting numbers are positive integers {1, 2, 3, ...}; some counting number y is said to divide or be a factor of some counting number x if $x/y = z$ where z is a counting number; common factors of two terms are counting numbers greater than 1 which divide both terms; two numbers are said to be relatively prime if they are counting numbers having no common factors. For example, 4 and 35 are relatively prime, as are 1 and 10, while 4 and 26 are not.

Proof by contradiction: Assume that $\sqrt{p/q}$ is rational, where p and q are relatively prime and at most one of them is a perfect square. Then we can write

$$\sqrt{\frac{p}{q}} = \frac{x}{y} \tag{1}$$

where x and y are relatively prime. But this last clause requires that x^2 and y^2 must also be relatively prime, because any prime factor of x^2 is also a prime factor of x and vice-versa, and likewise for y^2 and y. (Careful! A composite factor of x^2 need not be a factor of x. For example, 50 is a factor of 100 but not of 10, but 5 and 2 are factors of both.) We will find a contradiction to this requirement.

To do so, we start by rearranging Eq. (1) to get

$$py^2 = qx^2. \tag{2}$$

Thus, q divides py^2. But q does not divide p by assumption. Therefore, q divides y^2 and we can write

$$y^2 = nq \tag{3}$$

where n is some counting number. By the same argument, we can also conclude that

$$x^2 = mp. \tag{4}$$

Substituting Eqs. (3) and (4) into (2) and simplifying shows that

$$m = n \equiv N. \tag{5}$$

But N cannot be equal to 1, because if it were, then Eqs. (3) and (4) would imply that $p = x^2$ and $q = y^2$ so that both are perfect squares, which we assumed not to be the case. We now have a contradiction because we see from Eqs. (3) and (4) that N is a common factor of x^2 and y^2.

Comment: This proves for example that $\sqrt{3}$ is irrational!