Postage Stamps Problem—C.E. Mungan, Fall 2016

You have an unlimited supply of each of two (positive integer) denominations of postage stamps: *a* cents and *b* cents, where *a* and *b* are relatively prime. You can make various total postage amounts by putting some of each kind of stamp on a package. (Call these "makeable" amounts.) Clearly there are certain totals you cannot make. (Call them "unmakeable" amounts.) Prove that there is a maximum unmakeable amount *U* (i.e., you can make any amount larger than it) and find a formula for *U*.

Note that if *a* and *b* have a common factor, say 3, then any linear combination $pa+qb$ will also be divisible by 3. In that case, it will never be possible to achieve certain arbitrarily large amounts such as primes or 10^n for any integer *n*. This observation explains why *a* and *b* have to be relatively prime to have at least a chance of success.

Perhaps the best way to tackle this problem is to start by playing around with various specific values for *a* and *b*. Without loss of generality we can assume that $b > a$. The original problem (cf. bottom of http://www-groups.dcs.st-and.ac.uk/history/Biographies/Sylvester.html) suggested the values $a = 5$ and $b = 17$. You can quickly convince yourself that $U = 63$. By similarly playing with other values, I hypothesize that $U = ab - a - b$. Note that since *a* does not divide *b*, *a* cannot be equal to 1, and thus *a* and *b* can be no smaller than 2 and 3, respectively. Thus $a - 1 \geq 1$. Consequently $b > a$ implies $b(a-1) > a$ and it follows that *U* is necessarily positive.

It turns out that the key to this problem is the following lemma: If you compute the remainders of $\{b/a, 2b/a, 3b/a, \ldots, (a-1)b/a\}$ you get the values $\{1, 2, 3, \ldots, a-1\}$ after rearranging them into ascending order. To prove this lemma, first suppose that the quotient of b/a truncated to the next lowest integer is *Q* and the remainder is *R*, such that $b = aQ + R$ where $1 \leq R \leq a-1$. In that case, $2b = a(2Q) + (2R)$ and likewise for 3*b*, 4*b*, and so on. If $R = 1$, then $2R = 2$, $3R = 3$, and the proof is complete. If instead $R \geq 2$ then as you compute 2*R*, 3*R*, and so on you will eventually hit a value larger than *a* which is not the remainder unless you subtract *a* from it. Thus we compute this remainder as $NR - a$ where *N* is a (positive) integer. But it is impossible for this result to duplicate some previous remainder PR (where *P* is a positive integer smaller than *N*) because *a* is not divisible by *R*. (Otherwise $b = aQ + R$ would also be divisible by *R*, contradicting the fact that *a* and *b* are relatively prime.) Thus $NR - a$ must instead be a unique new remainder. It is clear that similar logic will work in general, so let's now accept the lemma as being proven. (Just as an example to show how the list of remainders works out, try $a = 7$ and $b = 17$ to get remainders of 3, 6, 2, 5, 1, and 4 in the order of the first list above. In fact, there is a systematic order of the remainders in general, determined by the value of the first remainder.)

Okay we can now solve the stamps problem! Let $n = a - \text{remainder}(mb/a)$ where *m* ranges over the integers from 0 to $a-1$. By our lemma, *n* can be rearranged into the values $\{1, 2, 3, \ldots, a\}$. Now observe that $mb + n$ is, by construction, divisible by *a*. So let $mb + n = Ma$ where *M* is some positive integer. Next consider the integer $S + n = S + Ma - mb$ where *S* is any makeable starting value you like. That is, *S* is expressible as a linear combination of *a* and *b*, say $S = Aa + Bb$. Then $S + n = (A + M)a + (B - m)b$ is also makeable *provided* that $B \geq m$ (because we cannot use a negative number of stamps of denomination *b*). This condition will be ensured for all values of *m* provided that $B \geq a - 1$. Choosing $B = a - 1$ and $A = 0$, we can thus make the definite amount $S = (a-1)b = ab - b$ as well as the amounts $S + 1$, $S + 2$, …, $S + a$ by letting *n* run from 1 up to a. Next choose $B = a-1$ and $A = 1$ to see that you can also make the next successive a integer amounts up to $S + 2a$. Thus you can bootstrap yourself along to make any amount equal to or greater than $ab - b$ that you like.

Finally let's work downward instead, starting from any makeable S and subtracting n from it to get $S - n = (A - M)a + (B + m)b$. This result will be makeable provided that $A \geq M$. The largest possible value of n is a so let's adopt it to get down to the lowest possible amount we can. We see from our definition of n that that implies $m = 0$ in which case $M = 1$. Thus we must have $A \geq 1$, which implies that values of $S = Aa + Bb$ smaller than ab cannot be a multiple of b (because that would imply $A = 0$). Our goal is now to find as small a makeable value of S that we can that is not a multiple of b. Well, we already know from the previous paragraph that all values of S greater than or equal to $(a-1)b$ are makeable. But by inspection, $(a-1)b$ is a multiple of b. Consequently $(a-1)b - a = ab - a - b$ is unmakeable. So next let's try $(a-1)b + 1$. By inspection, it is not a multiple of b. Choosing it to be S, we conclude that we can make $(a-1)b + 1 - a = (a-1)(b-1)$ and the proof is complete.

It is amusing to consider some extensions of this problem:

(1) Some total amounts can be made more than way. (For example, for $a = 2$ and $b = 3$, there are three different combinations of stamps to make a total of 12.) Other totals can only be made one way. (For the same values of a and b as in the previous example, there is only one way to make 13.) I hypothesize that *all* multiples and *only* multiples of ab can be made more than one way.

(2) If the preceding hypothesis is correct, then there is only one way to make each possible amount larger than ab but smaller than $2ab$. If you want to make one of each of them, you will thus need some definite number X of the a stamps, and Y of the b stamps. Do these two values have any special properties? For example, are there simple formulas for them? Are they relatively prime? As a starting point, notice that the money $Xa + Yb$ you have to spend to buy all these stamps must equal the sum of the total amounts, $3ab(ab-1)/2$, and thus you can compute Y if you know X.

<u>Addendum</u>: Bob Siddon has come up with a proof of a simple formula for all of the unmakeable amounts: $U(\alpha,\beta) = ab - \alpha a - \beta b$ where α and β are any positive integers such that U is positive. For example, if a is 5 cents and b is 17 cents, then the largest unmakeable amount is $\alpha = \beta = 1 \Rightarrow U = 63$ cents. If you instead choose say $\alpha = 10$ and $\beta = 2$, then you get $U = 1$ cent, which is obviously also unmakeable. It is nifty that such a simple formula exists for the unmakeable amounts. You could use it to count how many unmakeable amounts there are for arbitrary values of a and b, as one application among many.