Derivations of Stirling's Approximation—C.E. Mungan, Spring 1998

Method 1: By Taylor Series

Begin with

$$n! = \Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \int_0^\infty e^{n\ln x - x} dx$$

and for convenience define $f(x) = n\ln x - x$. By graphing $f(x)$, you can convince yourself that it peaks at $x = n$ since $f'(n) = 0$. Hence, we can approximate $f(x)$ by expanding it around $x = n$. (By expanding around the maximum, the integral will include most of the area.) We get

$$f(x) = f(n) + f'(n)\cdot(x-n) + \tfrac{1}{2} f''(n)\cdot(x-n)^2 + \cdots$$

$$\Rightarrow e^{f(x)} = n^n e^{-n} \times e^{-(x-n)^2/2n} \times \cdots$$

since $f''(n) = -1/n$. Recalling the normalization of a Gaussian of mean n and standard deviation $\sqrt{n}$, we immediately obtain

$$n! \sim n^n e^{-n} \int_0^\infty e^{-(x-n)^2/2(\sqrt{n})^2} dx = n^n e^{-n} \sqrt{2\pi n}$$

which is Boas Eq. (11.11.1). I extended the lower limit in evaluating the integral to $-\infty$ since n is assumed to be large.

Method 2: Using Infinite Series, Products, and Limits

This method due to Mermin is longer, but enlightening. By inspection we see that

$$n! = 1\times 2\times 3\times\cdots\times(n-1)\times n$$

$$= \left(\tfrac{1}{2}\right)^{3/2} \times \left(\tfrac{2}{3}\right)^{5/2} \times \left(\tfrac{3}{4}\right)^{7/2} \times\cdots\times \left(\tfrac{n-1}{n}\right)^{n-1/2} \times n^{n+1/2}$$

$$= \frac{n^{n+1/2}}{\prod_{m=1}^{n-1}\left(1+\tfrac{1}{m}\right)^{m+1/2}} = \frac{n^{n+1/2}}{e^{n-1}} \frac{\prod_{m=n}^{\infty}\left(1+\tfrac{1}{m}\right)^{m+1/2}/e}{\prod_{m=1}^{\infty}\left(1+\tfrac{1}{m}\right)^{m+1/2}/e}$$

Let's define a constant P such that

$$\prod_{m=1}^{\infty}\left(1+\tfrac{1}{m}\right)^{m+1/2}/e \equiv \frac{e}{\sqrt{2P}} \qquad (1a)$$

which is meaningful because the infinite product can easily be shown to converge using Eq. (1b) below. It follows that

$$n! = n^n e^{-n}\sqrt{2Pn} \prod_{m=n}^{\infty}\left(1+\tfrac{1}{m}\right)^{m+1/2}/e \qquad (2a)$$

and we can show that the infinite product in this equation is negligible for large n. In particular, each term in this product has m large and

$$\left(1+\tfrac{1}{m}\right)^{m+1/2}/e = \exp\left[\left(m+\tfrac{1}{2}\right)\ln\left(1+\tfrac{1}{m}\right)-1\right]$$
$$= \exp\left[\left(m+\tfrac{1}{2}\right)\left(\tfrac{1}{m}-\tfrac{1}{2m^2}+\tfrac{1}{3m^3}-\cdots\right)-1\right] \qquad (1b)$$
$$= \exp\left[\tfrac{1}{12m^2}-\tfrac{1}{12m^3}+\cdots\right]$$

where use was made of Boas Eq. (1.13.4) in the second step, which applies since it is always true that $\tfrac{1}{m} \le 1$. But for sufficiently large m, the argument of the exponential is negligibly small and it follows that

$$\boxed{\lim_{m\to\infty}\left(1+\tfrac{1}{m}\right)^{m+1/2} = e}$$

which actually converges quite rapidly (e.g., try it on a calculator for m = 1, 10, 100) and is in and of itself a very useful result. Thus, Eq. (2a) becomes

$$n! \sim n^n e^{-n}\sqrt{2Pn} \qquad (2b)$$

even for merely moderately large n.

The last and hardest step is to prove that $P = \pi$, so that Eq. (2b) gives the desired result. For this purpose, we first derive Wallis' formula for π starting from

$$\lim_{n\to\infty}\frac{\int_0^{\pi/2}\sin^{2n}\theta\,d\theta}{\int_0^{\pi/2}\sin^{2n+1}\theta\,d\theta} = \lim_{n\to\infty}\frac{\int_0^{\pi/2}\sin^{2n}\theta\,d\theta}{\int_0^{\pi/2}\sin^{2n}\theta\cos\left(\tfrac{\pi}{2}-\theta\right)d\theta} = 1$$

since for large n, $\sin^{2n}\theta$ is nonzero on (0,π/2) only for θ near π/2, in which case $\cos\left(\tfrac{\pi}{2}-\theta\right)\approx 1$. However,

$$\frac{\int_0^{\pi/2}\sin^{2n}\theta\,d\theta}{\int_0^{\pi/2}\sin^{2n+1}\theta\,d\theta} = \frac{\tfrac{1}{2}B(n+\tfrac{1}{2},\tfrac{1}{2})}{\tfrac{1}{2}B(n+1,\tfrac{1}{2})} = \frac{\dfrac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n+1)}}{\dfrac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}}$$
$$= \left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2(n+\tfrac{1}{2}) = \left[\frac{1\times3\times5\times\cdots\times(2n-1)}{2^n 1\times2\times3\times\cdots\times n}\sqrt{\pi}\right]^2(n+\tfrac{1}{2})$$
$$= \frac{\pi}{2}\,\frac{1\times3\times5\times\cdots\times(2n-1)}{2\times4\times6\times\cdots\times2n}\,\frac{3\times5\times7\times\cdots\times(2n+1)}{2\times4\times6\times\cdots\times2n}$$

using the intermediate result of Boas Problem 11.5.1. Hence, in the limit as $n\to\infty$ we have

$$\boxed{\frac{\pi}{2} = \frac{2\cdot2}{1\cdot3}\times\frac{4\cdot4}{3\cdot5}\times\frac{6\cdot6}{5\cdot7}\times\cdots}$$

which is Wallis' formula. Finally, consider the following quantity

$$2n\left[\frac{(2n)!}{(2^n n!)^2}\right]^2 = 2n\left[\frac{1\times2\times3\times\cdots\times2n}{(2\times4\times6\times\cdots\times2n)^2}\right]^2 = 2n\left[\frac{1\times3\times5\times\cdots\times(2n-1)}{2\times4\times6\times\cdots\times2n}\right]^2$$

$$= \left[\frac{2\cdot2\times4\cdot4\times6\cdot6\times\cdots\times(2n-2)\cdot(2n-2)\times2n}{1\cdot3\times3\cdot5\times5\cdot7\times\cdots\times(2n-3)\cdot(2n-1)\times(2n-1)}\right]^{-1}$$

so that

$$\lim_{n\to\infty} 2n\left[\frac{(2n)!}{(2^n n!)^2}\right]^2 = \frac{2}{\pi}$$

using Wallis' formula. (Compare Boas Problem 11.11.4.) On the other hand, Eq. (2b) implies

$$2n\left[\frac{(2n)!}{(2^n n!)^2}\right]^2 \sim 2n\frac{(2n)^{4n}e^{-4n}4Pn}{2^{4n}n^{4n}e^{-4n}4P^2n^2} = \frac{2}{P}$$

for large n, and by comparing the last two equations we see that we must have $P = \pi$ as required. See Mermin for a discussion of how we can go on to refine Stirling's approximation by substituting Eq. (1b) into Eq. (2a).