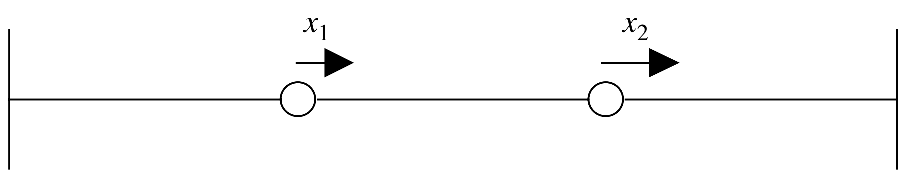
Potential Energy of Stretched Springs—C.E. Mungan, Fall 2003

Suppose that we join a set of point masses into a line interconnected by springs, as in the usual coupled oscillators configuration. The two end springs will be attached to fixed walls, between which the whole arrangement will be stretched. We wish to find expressions for the elastic potential energy of the system, both for longitudinal displacements of the masses (i.e., along the line of the springs) and for transverse displacements (i.e., perpendicular to the line of the springs). For simplicity, I will specifically consider the case of three identical springs. The generalization to inequivalent springs and to a different number of springs will be clear in the end.

Let's begin with the more common longitudinal case sketched below.



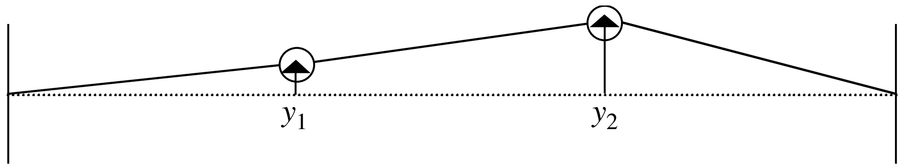
Each spring has stiffness constant k, relaxed length l, and equilibrium length $L = l + u$, where u is the amount by which each spring had to be initially stretched (or compressed, if u is negative) to get the arrangement to extend from the left wall to the right wall. Next consider an arbitrary longitudinal displacements of the two masses by x_1 and x_2, as drawn. Since the potential energy of a spring is determined by the total stretch of the spring from its relaxed length, the elastic potential energy of the system is thus found to be

$$\begin{aligned} U &= \tfrac{1}{2}k(x_1 + u)^2 + \tfrac{1}{2}k(x_2 - x_1 + u)^2 + \tfrac{1}{2}k(-x_2 + u)^2 \\ &= U_0 + \tfrac{1}{2}kx_1^2 + \tfrac{1}{2}k(x_2 - x_1)^2 + \tfrac{1}{2}kx_2^2 \end{aligned} \tag{1}$$

where the second line follows from expanding the squares in the first line, and where U_0 is a constant equal to $3ku^2/2$, as can be obtained by putting $x_1 = x_2 = 0$ in the first line. That is, U_0 has the physical meaning of the potential energy at equilibrium, which we can conveniently redefine as our zero-energy reference configuration, so that $U_0 = 0$. We therefore conclude that the potential energy of a longitudinally displaced set of n springs can be written as

$$\boxed{U = \sum_{i=1}^{n} \tfrac{1}{2}k_i\delta_{ix}^2} \tag{2}$$

where k_i is the stiffness constant of the i-th spring and δ_{ix} is the difference in the longitudinal displacements of the two ends of the i-th spring from their equilibrium positions (i.e., it is the stretch of the spring compared to its *equilibrium* length). It is important to note that Eq. (2) is *exact* (assuming Hookean springs) regardless of the values of x_i, l, and u. This is not true of the transverse configuration, which is diagrammed below.



In this case the elastic potential energy of the system is

$$U = \tfrac{1}{2}k\left[\sqrt{L^2 + y_1^2} - l\right]^2 + \tfrac{1}{2}k\left[\sqrt{L^2 + (y_2 - y_1)^2} - l\right]^2 + \tfrac{1}{2}k\left[\sqrt{L^2 + y_2^2} - l\right]^2 \tag{3}$$

since the terms in the square brackets are the stretches of the springs from their relaxed length l. If we assume the transverse displacements y_1 and y_2 are both small compared to the equilibrium stretch $L - l \equiv u$ of each spring, then we can Taylor expand first the square roots and then the squares as follows,

$$\begin{aligned}
U &= \tfrac{1}{2}k\left[L\left(1 + \frac{y_1^2}{L^2}\right)^{1/2} - l\right]^2 + \tfrac{1}{2}k\left[L\left(1 + \frac{\{y_2 - y_1\}^2}{L^2}\right)^{1/2} - l\right]^2 + \tfrac{1}{2}k\left[L\left(1 + \frac{y_2^2}{L^2}\right)^{1/2} - l\right]^2 \\
&\cong \tfrac{1}{2}k\left[L\left(1 + \frac{y_1^2}{2L^2}\right) - l\right]^2 + \tfrac{1}{2}k\left[L\left(1 + \frac{\{y_2 - y_1\}^2}{2L^2}\right) - l\right]^2 + \tfrac{1}{2}k\left[L\left(1 + \frac{y_2^2}{2L^2}\right) - l\right]^2 \\
&= \tfrac{1}{2}k\left[u + \frac{y_1^2}{2L}\right]^2 + \tfrac{1}{2}k\left[u + \frac{(y_2 - y_1)^2}{2L}\right]^2 + \tfrac{1}{2}k\left[u + \frac{y_2^2}{2L}\right]^2 \\
&= \tfrac{1}{2}ku^2\left[1 + \frac{y_1^2}{2uL}\right]^2 + \tfrac{1}{2}ku^2\left[1 + \frac{(y_2 - y_1)^2}{2uL}\right]^2 + \tfrac{1}{2}ku^2\left[1 + \frac{y_2^2}{2uL}\right]^2 \\
&\cong \tfrac{1}{2}ku^2\left[1 + \frac{y_1^2}{uL}\right] + \tfrac{1}{2}ku^2\left[1 + \frac{(y_2 - y_1)^2}{uL}\right] + \tfrac{1}{2}ku^2\left[1 + \frac{y_2^2}{uL}\right] \\
&= \tfrac{3}{2}ku^2 + \tfrac{1}{2}k\frac{u}{L}\left[y_1^2 + (y_2 - y_1)^2 + y_2^2\right].
\end{aligned} \tag{4}$$

We again redefine the equilibrium energy $U_0 = 3ku^2/2$ to be zero, since it is merely an uninteresting additive constant. Finally if the springs start out highly stretched ($u >> l$), then $u/L = u/(u + l) \approx 1$, and Eq. (4) becomes, when generalized,

$$\boxed{U = \sum_{i=1}^{n} \tfrac{1}{2}k_i\delta_{iy}^2} \tag{5}$$

where δ_{iy} is the difference in the transverse displacements of the two ends of the i-th spring from their equilibrium positions (or equivalently, to within an uninteresting constant, δ_{iy}^2 is the square of the *total* length of the i-th spring). But it is important to notice that Eq. (5) is only valid provided that both y_i and l are small compared to u, UNLESS all of the springs have zero relaxed length ($l = 0$), in which case we see from Eq. (3) that Eq. (5) is exact for any values of y_i and u.

In contrast, note from Eq. (4) that $U = 0$ to quadratic order if the springs start out unstretched ($u = 0$). This makes sense intuitively because small transverse displacements hardly stretch these floppy springs.