Sum of Integers Cubed—C.E. Mungan, Spring 2021

Show that the sum of the first n integers cubed is a quartic polynomial in n.

One way to solve this kind of problem is to follow these three steps:

1. Try some sample values of n to guess a formula for the pattern.
2. Prove that the formula is correct by induction.
3. Rearrange the formula to get the requested form for the final answer.

Let's illustrate these steps for the current problem.

Here are the sums for the first few values of n:

$$1^3 = 1 = 1^2$$
$$1^3 + 2^3 = 9 = 3^2$$
$$1^3 + 2^3 + 3^3 = 36 = 6^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2$$

and so it appears likely that the formula is the square of the sum of the first n integers, so that

$$\boxed{\sum_{m=1}^{n} m^3 = \left[\frac{n(n+1)}{2}\right]^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2} \tag{1}$$

which is the desired quartic polynomial. We prove Eq. (1) by induction. First verify it for the starting value $n = 1$, as follows from

$$1^3 = \left[\frac{1(2)}{2}\right]^2 \quad \text{or equivalently} \quad 1^3 = \frac{1}{4}1^4 + \frac{1}{2}1^3 + \frac{1}{4}1^2 . \tag{2}$$

Next assume it holds for $n-1$,

$$\sum_{m=1}^{n-1} m^3 = \left[\frac{(n-1)n}{2}\right]^2 , \tag{3}$$

and use Eq. (3) to prove Eq. (1) according to

$$\sum_{m=1}^{n} m^3 = \left[\frac{(n-1)n}{2}\right]^2 + n^3 = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 . \tag{4}$$

Q.E.D.