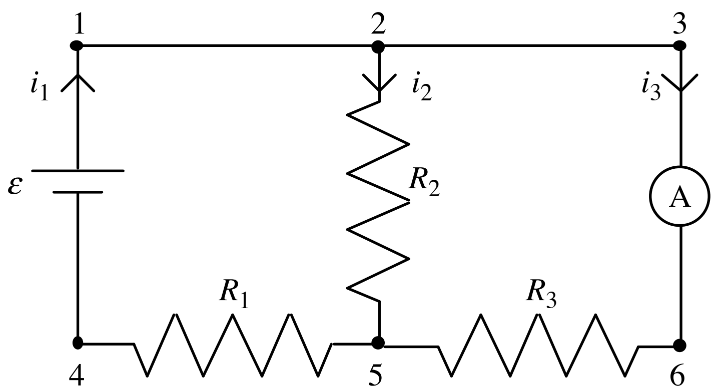
A Surprising Circuit Symmetry—C.E. Mungan, Spring 2014

The following problem is taken from the 9th edition of HRW, number 49(b) at the end of chapter 27. Show that the ammeter reading in the following circuit is unchanged if one interchanges the battery and the ammeter. All elements are assumed to be ideal.



It suffices to show that i_3 is unchanged if R_1 and R_3 are interchanged. To prove that, I will show in three different ways that

$$i_3 = \frac{\varepsilon R_2}{R_1 R_2 + R_2 R_3 + R_1 R_3} \tag{1}$$

which by inspection satisfies what we want to show.

Method #1: Kirchhoff analysis

Since we want i_3, let's apply KVL to the right-hand loop 2365 to get

$$-i_3 R_3 + i_2 R_2 = 0 \quad \Rightarrow \quad i_2 = i_3 \frac{R_3}{R_2} \tag{2}$$

and to the outer loop 4136 to find

$$\varepsilon - i_3 R_3 - i_1 R_1 = 0 \quad \Rightarrow \quad i_1 = \frac{\varepsilon}{R_1} - i_3 \frac{R_3}{R_1}. \tag{3}$$

Finally, apply KCJ to the top junction 2 to obtain

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad i_3 = i_1 - i_2. \tag{4}$$

Substitute Eqs. (2) and (3) into (4) to deduce

$$i_3 = \frac{\varepsilon}{R_1} - i_3 \frac{R_3}{R_1} - i_3 \frac{R_3}{R_2} \tag{5}$$

which rearranges into Eq. (1).

Method #2: parallel/series resistor rules

Resistors R_2 and R_3 are in parallel so that

$$R_{23} = \frac{R_2 R_3}{R_2 + R_3} \tag{6}$$

which we put in series with R_1 to get the total equivalent circuit resistance of

$$R_{\text{eq}} = R_1 + R_{23} \tag{7}$$

so that

$$i_1 = \frac{\varepsilon}{R_1 + R_{23}}\,. \tag{8}$$

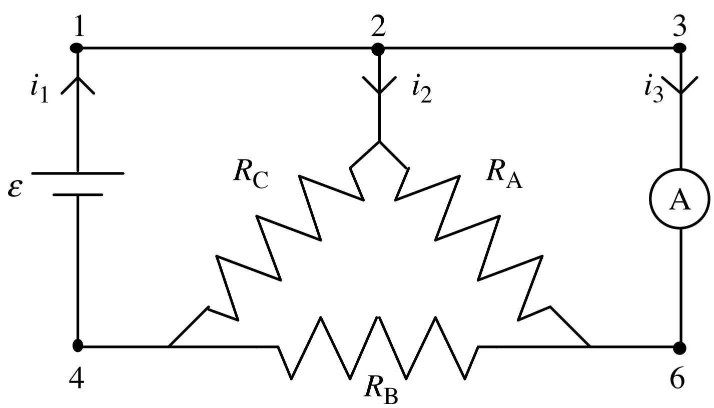
However, the voltages across resistors R_3 and R_{23} are equal, and thus

$$i_3 R_3 = i_1 R_{23} \quad \Rightarrow \quad i_1 = i_3 \frac{R_3}{R_{23}}\,. \tag{9}$$

Equate the right-hand sides of Eqs. (8) and (9) to get (1) after a bit of algebra.

Method #3: delta-star transform

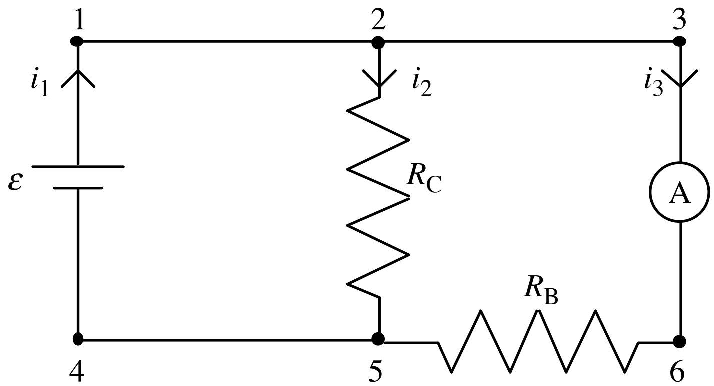
A delta-star transform replaces the three star-arranged resistors $\{R_1, R_2, R_3\}$ with three delta-arranged resistors $\{R_\text{A}, R_\text{B}, R_\text{C}\}$ so that the circuit becomes as sketched below.



Here

$$R_\text{B} = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_2} \tag{10}$$

since R_2 is the resistor opposite R_B. However, the ammeter now shorts out R_A so that this circuit is equivalent to the one drawn on the top of the next page.



One sees from this revised circuit that

$$\varepsilon = i_3 R_B \tag{11}$$

which by inspection rearranges into Eq. (1).

Closing comment

Noting that the middle and right branches of the original circuit are equivalent if one interchanges subscripts 2 and 3, then Eq. (1) implies that

$$i_2 = \frac{\varepsilon R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}, \tag{12}$$

in agreement with Eq. (2). Substituting Eqs. (1) and (12) into (4) gives

$$i_1 = \varepsilon \frac{R_2 + R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3}. \tag{13}$$

Therefore only i_3 remains unchanged when one interchanges the battery and ammeter (by interchanging R_1 and R_3). In contrast, both i_2 and i_1 change when we do so.