**Shooting at a Constant-Velocity Target—C.E. Mungan, Fall 2011**

Suppose a target is initially located at position $\mathbf{r}_{T0}$ and is moving with constant velocity $\boldsymbol{\upsilon}_T$. You have at your disposal a gun located at position $\mathbf{r}_{B0}$ that fires bullets at constant speed $\upsilon_B$ in any direction $\hat{\mathbf{B}}$ of your choosing. Assuming you can hit the target, in which direction should you fire and at what time $t$ will the bullet hit it?

As a function of time, the target has position

$$\mathbf{r}_T = \mathbf{r}_{T0} + \boldsymbol{\upsilon}_T t \tag{1}$$

while the bullet has position

$$\mathbf{r}_B = \mathbf{r}_{B0} + \boldsymbol{\upsilon}_B t \; . \tag{2}$$

To get a hit, we require the two to have the same position. Equating the right-hand sides of Eqs. (1) and (2), and defining the initial position of the target relative to the gun to be $\mathbf{D} \equiv \mathbf{r}_{T0} - \mathbf{r}_{B0}$, leads to

$$\mathbf{D} = (\boldsymbol{\upsilon}_B - \boldsymbol{\upsilon}_T) t \; . \tag{3}$$

The three components of this equation are

$$D_x = (\upsilon_{Bx} - \upsilon_{Tx}) t \; , \tag{4a}$$

$$D_y = (\upsilon_{By} - \upsilon_{Ty}) t \; , \tag{4b}$$

and

$$D_z = (\upsilon_{Bz} - \upsilon_{Tz}) t \; . \tag{4c}$$

Solve Eqs. (4) for the components of $\boldsymbol{\upsilon}_B$, square them, and add them together to get

$$\left( \frac{D_x}{t} + \upsilon_{Tx} \right)^2 + \left( \frac{D_y}{t} + \upsilon_{Ty} \right)^2 + \left( \frac{D_z}{t} + \upsilon_{Tz} \right)^2 = \upsilon_B^2 \; . \tag{5}$$

Expand each square, collect like terms together, and multiply through by $t^2$ to end up with

$$(\upsilon_B^2 - \upsilon_T^2) t^2 - 2(\boldsymbol{\upsilon}_T \bullet \mathbf{D}) t - D^2 = 0 \; . \tag{6}$$

Assuming the bullet has higher speed than the target, we choose the positive solution of the quadratic equation to find the time it hits as

$$t = \frac{\boldsymbol{\upsilon}_T \bullet \mathbf{D} + \sqrt{(\boldsymbol{\upsilon}_T \bullet \mathbf{D})^2 + (\upsilon_B^2 - \upsilon_T^2) D^2}}{\upsilon_B^2 - \upsilon_T^2} \; . \tag{7}$$

Knowing the time, we substitute it back into Eq. (3) to get the firing angle,

$$\hat{\mathbf{B}} = \frac{\mathbf{D}/t + \boldsymbol{\upsilon}_{\mathrm{T}}}{\upsilon_{\mathrm{B}}} \,. \tag{8}$$