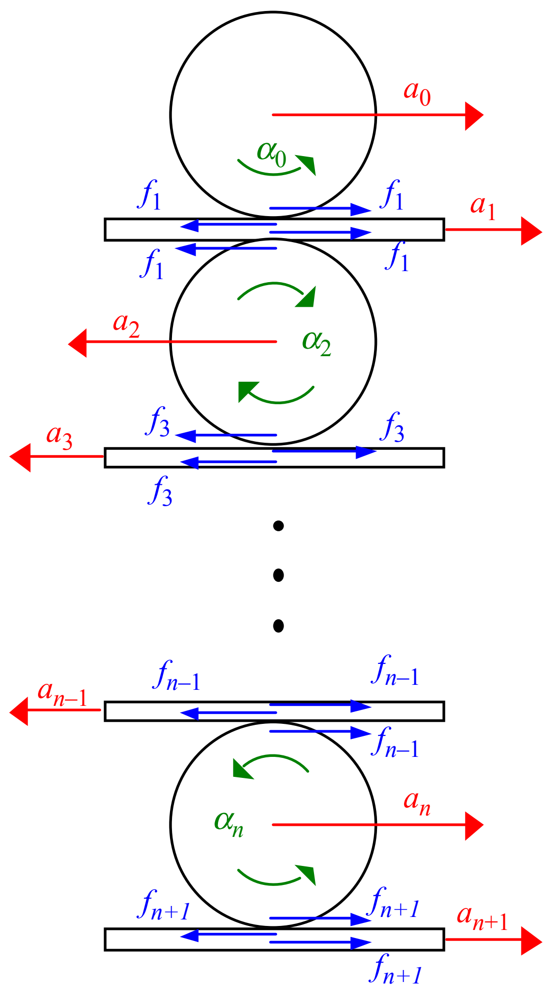
Infinite Tower of Alternating Sheets and Nonslipping Rollers—C.E. Mungan, Fall 2021

A uniform solid cylinder of mass M and radius R is placed on top of a flat massless sheet. The axis of the cylinder is into the page. The coefficient of static friction between the cylinder and the sheet is large enough that the cylinder can roll leftward or rightward along the sheet without slipping. Label the cylinder as object 0 and the sheet as object 1 which together form the $\{0,1\}$ layer as shown at the top of the following diagram. We then place that layer above an identical $\{2,3\}$ layer consisting of cylinder 2 and sheet 3, as also shown. Next put the combination on top of another layer $\{4,5\}$ which is not shown and repeat by proceeding downward for an infinite number of layers. An arbitrary layer is denoted $\{n,n+1\}$ where n is a non-negative even integer denoting a cylinder. That arbitrary layer is also shown in the diagram, along with the sheet $n-1$ just above it that again rolls without slipping relative to cylinder n below it. (As we will see, the directions for the forces and accelerations of that layer are correct only if n is divisible by 4; for layers 2, 6, 10, … all of those directions must be reversed.)



We are now going to imagine that the top sheet 1 has a rightward acceleration a_1. Since it is in contact with cylinder 0 above it and cylinder 2 below it that both roll without slipping, there must be frictional forces between the sheet and each of these cylinders. Each of them come in action-reaction pairs: friction pointing one way acting on the bottom of cylinder 0 which is equal and opposite to the friction pointing the other way on the top of sheet 1, and likewise friction pointing one way acting on the top of cylinder 2 which is equal and opposite to the friction pointing the other way on the bottom of sheet 1. However, since the sheet is massless, the net force on it must be zero (by N2L) and thus the friction forces acting on the top and bottom of sheet 1 must be equal and opposite. Hence, all four of these forces must have the same magnitude, let us call it f_1 since they are due to interactions with sheet 1. (Likewise, the diagram shows three of the four frictional forces associated with sheet 3, three with sheet $n-1$, and three with sheet $n+1$.)

Returning to the top $\{0,1\}$ layer with sheet 1 accelerating rightward, cylinder 0 above it would slip leftward (if it started from rest) relative to sheet 1 in the absence of friction. Consequently f_1 on the bottom of cylinder 0 must point rightward. But f_1 is the net force on that cylinder and thus a_0 must point rightward. In addition, that force produces a ccw torque about the center of cylinder 0 which will result in the indicated ccw angular acceleration α_0 of the cylinder. We now understand how the directions of all the forces and accelerations of layer $\{0,1\}$ have been assigned in the diagram. Notice that cylinder 0 has "back" spin: its center is translating rightward but it is rotating ccw. This could not happen if the cylinder were rolling without slipping on a stationary floor. The only reason it can happen here is that $a_1 > a_0$ as we will soon show mathematically.

Consider the arbitrary layer $\{n, n+1\}$ shown in the diagram. The directions of a_n, α_n, and f_{n+1} have been assumed to match those of layer $\{0,1\}$. Now suppose we wanted to place sheet $n-1$ on top of this combination. One way we could do that is to let sheet $n+1$ get driven half a circumference around cylinder *n*. Consequently sheet $n-1$ must have a leftward acceleration (labeled a_{n-1} in the diagram). We conclude that alternate sheets must accelerate in opposite directions: a_1 is rightward, a_3 is leftward, a_5 is rightward, and so on. But that implies layer $\{2,3\}$ must have the directions of all of its forces and accelerations *reversed* compared to those of layer $\{0,1\}$.

To summarize, we now understand the labels and directions of all of the force and acceleration arrows in the diagram. However, the lengths of these arrows are not to scale; they serve only to indicate the positive directions but not the strengths of the accelerations and forces. We will next find those strengths from a Newtonian analysis. In particular, notice the frictional force on the bottom of any cylinder must be stronger than the frictional force on the top of the same cylinder, in order to give the indicated directions for the angular accelerations of each cylinder.

First consider the top $\{0,1\}$ layer. Newton's second law for translations of the center of mass of cylinder 0 becomes

$$f_1 = Ma_0 \tag{1}$$

whereas for rotations about the center of the cylinder in a reference frame accelerating rightward at a_0 one obtains

$$\alpha_0 = \frac{\tau}{I} \quad \Rightarrow \quad \frac{a_1 - a_0}{R} = \frac{f_1 R}{MR^2 / 2}. \tag{2}$$

If we choose units such that $M = 1$ then the solution of these two equations is

$$f_1 = a_0 \quad \text{and} \quad a_1 = 3a_0 \tag{3}$$

where we from now on assume that a_0 is given (rather than a_1 as in the preceding discussion).

For any subsequent layer $\{n, n+1\}$ where n is a non-negative even integer, we can find the translational accelerations a_n and a_{n+1} and the frictional force f_{n+1} recursively in terms of the known values a_{n-1} and f_{n-1} from the previous layer. [For example, we know a_1 and f_1 from Eq. (3) and we want to find a_2, a_3, and f_3 for the next layer down.] The analog of Eq. (1) is

$$f_{n+1} + f_{n-1} = Ma_n \quad \Rightarrow \quad f_{n+1} = a_n - f_{n-1} \tag{4}$$

in rescaled units. Likewise the analog of Eq. (2) at the contact point between sheet $n-1$ and cylinder n in the preceding diagram becomes

$$\alpha_n = \frac{a_{n-1} + a_n}{R} = \frac{(f_{n+1} - f_{n-1})R}{MR^2 / 2} \quad \Rightarrow \quad f_{n+1} = f_{n-1} + \frac{a_{n-1} + a_n}{2} \tag{5}$$

in a reference frame accelerating rightward at a_n. Equate the RHS of these last two numbered equations to deduce

$$a_n = a_{n-1} + 4f_{n-1} \tag{6}$$

and substitute that back into Eq. (4) to find

$$f_{n+1} = a_{n-1} + 3f_{n-1}. \tag{7}$$

Finally the analog of Eq. (5) at the contact point between sheet $n+1$ and cylinder n is

$$\alpha_n = \frac{a_{n+1} - a_n}{R} = \frac{f_{n+1} - f_{n-1}}{MR^2 / 2} \quad \Rightarrow \quad a_{n+1} = 3a_{n-1} + 8f_{n-1}. \tag{8}$$

One simple numerical approach is to input these three formulas into Excel and evaluate them to get the results shown in the table at the top of the next page (where accelerations are in units of a_0 and friction is in units of Ma_0) with the entries in the first row entered as starting values. For example, the acceleration of cylinder 4 is $41a_0$, the acceleration of sheet 3 is $17a_0$, and the friction associated with sheet 5 is $35Ma_0$. Alternatively it is straightforward to find explicit formulas for these three quantities (as a function of n) using Mathematica, as presented in the Appendix. The last column is the ratio of the acceleration of a cylinder to the acceleration of the sheet immediately under it. For example, the second ratio is $a_2 / a_3 = 7/17 \approx 0.411764706$. The ratio quickly converges to the value 0.414213562 which the Appendix confirms to be exactly $2^{1/2} - 1$ in the limit as $n \to \infty$. The values in the table grow rapidly large. However, if we start with reasonable values at some large layer number, then the values will instead shrink toward zero as we go up the layers. (Equivalently, a_0 starts with some infinitesimal value.)

cylinder	cyl accel	layer	layer accel	friction	ratio
0	1	**1**	3	1	0.333333333
2	7	**3**	17	6	0.411764706
4	41	**5**	99	35	0.414141414
6	239	**7**	577	204	0.414211438
8	1393	**9**	3363	1189	0.414213500
10	8119	**11**	19601	6930	0.414213561
12	47321	**13**	114243	40391	0.414213562
14	275807	**15**	665857	235416	0.414213562
16	1607521	**17**	3880899	1372105	0.414213562
18	9369319	**19**	22619537	7997214	0.414213562
20	54608393	**21**	131836323	46611179	0.414213562
22	318281039	**23**	768398401	271669860	0.414213562
24	1855077841	**25**	4478554083	1583407981	0.414213562

Appendix: Mathematica Solution

Explicit formulas for the recursion solution of Eqs. (6) to (8) using "RSolve" are

$$a_n = \frac{\left(\sqrt{2}+1\right)^{n+1} - \left(\sqrt{2}-1\right)^{n+1}}{2} \quad \text{for cylinder } n = 0,2,4,... \tag{A1}$$

and

$$a_n = \frac{\left(\sqrt{2}+1\right)^{n+1} + \left(\sqrt{2}-1\right)^{n+1}}{2} \quad \text{for layer } n = 1,3,5,... \tag{A2}$$

with

$$f_n = \frac{\left(\sqrt{2}+1\right)^{n+1} - \left(\sqrt{2}-1\right)^{n+1}}{4\sqrt{2}} \quad \text{associated with layer } n = 1,3,5,... \tag{A3}$$

which one can check correctly reproduces the values given in the preceding table. Thus the ratio of Eq. (A1) evaluated at n to Eq. (A2) evaluated at $n+1$ in the limit as $n \to \infty$ is

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\left(\sqrt{2}+1\right)^{n+1} - \left(\sqrt{2}-1\right)^{n+1}}{\left(\sqrt{2}+1\right)^{n+2} + \left(\sqrt{2}-1\right)^{n+2}} = \lim_{n\to\infty} \frac{\left(\sqrt{2}+1\right)^{n+1} - 0}{\left(\sqrt{2}+1\right)^{n+2} + 0} = \frac{1}{\sqrt{2}+1} = \sqrt{2}-1\,. \tag{A4}$$