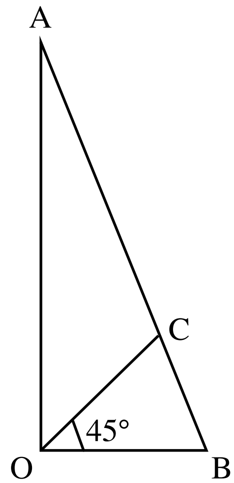
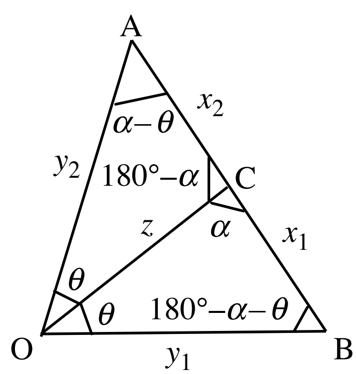
Triangle Bisector Theorem—C.E. Mungan, Spring 2016

Consider an arbitrary triangle AOB. Suppose the angle at vertex O is bisected with a line that intersects side AB at point C. How do the lengths of AC and CB compare to each other?

One's initial inclination might be to suppose they are equal. But by considering say a right triangle with the 90° angle at vertex O, the following diagram clearly shows they are unequal.



The general case is in the next diagram. The two equal angles after bisection are labeled θ and must obviously be strictly less than 90° because a triangle's internal angle BOA cannot exceed 180°. Likewise angle OCB is labeled α and is assumed to be less than or equal to 90° because if it is not, then angle OCA must be and one can simply exchange vertex labels A and B. By labeling all of the remaining internal angles in the figure, one sees that angle OAC equals $\alpha - \theta$ which must be positive. We conclude that the angles are restricted to the values $0° < \theta < \alpha \le 90°$.



From the law of sines we see that for triangle BOC

$$\frac{x_1}{\sin\theta} = \frac{y_1}{\sin\alpha} \quad \Rightarrow \quad \frac{x_1}{y_1} = \frac{\sin\theta}{\sin\alpha} \tag{1}$$

and for triangle AOC

$$\frac{x_2}{\sin\theta} = \frac{y_2}{\sin(180° - \alpha)} \quad \Rightarrow \quad \frac{x_2}{y_2} = \frac{\sin\theta}{\sin\alpha}\,. \tag{2}$$

Equating Eqs. (1) and (2) we obtain the triangle bisector theorem

$$\boxed{\frac{x_1}{x_2} = \frac{y_1}{y_2}} \tag{3}$$

so that lengths AC and CB are equal only if triangle AOB is isoceles with vertex angle at O. That in turn implies that angle α equals 90° and suggests another way to look at things. We can alternatively write the law of sines for triangle BOC as

$$\frac{x_1}{\sin\theta} = \frac{z}{\sin(180° - \alpha - \theta)} \quad \Rightarrow \quad x_1 = \frac{z\sin\theta}{\sin\alpha\cos\theta + \cos\alpha\sin\theta} \tag{4}$$

and for triangle AOC as

$$\frac{x_2}{\sin\theta} = \frac{z}{\sin(\alpha - \theta)} \quad \Rightarrow \quad x_2 = \frac{z\sin\theta}{\sin\alpha\cos\theta - \cos\alpha\sin\theta}\,. \tag{5}$$

Dividing Eq. (4) by (5) results in

$$\frac{x_1}{x_2} = \frac{\sin\alpha\cos\theta - \cos\alpha\sin\theta}{\sin\alpha\cos\theta + \cos\alpha\sin\theta} = \boxed{\frac{\tan\alpha - \tan\theta}{\tan\alpha + \tan\theta}}\,. \tag{6}$$

The middle expression shows that lengths AC and CB are indeed equal if $\alpha = 90°$. For any other angles α and θ in the ranges $0° < \theta < \alpha \le 90°$ it must be the case that length CB is smaller than length AC, as is true in the first diagram above for example.