Two Derivations of the Trig Functions of 15° and 75°—C.E. Mungan, Summer 2021

Two methods are presented here that can be used in class. The first method assumes that students know the cosine of 30°. (If not, see the second method to get that.) Start from

$$e^{i2\theta} = (e^{i\theta})^2 , \tag{1}$$

substitute the Euler identity into both sides, and take the real parts to get

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 \tag{2}$$

using the Pythagorean identity in the second step. Thus the half-angle formula for cosine is

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} . \tag{3}$$

Apply it to 15° to get

$$\cos 15° = \sqrt{\frac{1+\cos 30°}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \sqrt{\frac{\frac{1}{2}(1+\sqrt{3})^2}{4}} = \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \sin 75° . \tag{4}$$

Finally use the Pythagorean identity to obtain

$$\cos 75° = \sqrt{1-\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)^2} = \sqrt{\frac{4-2\sqrt{3}}{8}} = \boxed{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \sin 15° . \tag{5}$$

Ratios of Eqs. (4) and (5) give the tangents and cotangents of 15° and 75°. Reciprocals of each of the two equations give the secants and cosecants.

The second method is entirely graphical which I find appealing. The trick is to draw what is known as an Ailles rectangle (named after its discovery by a high school teacher). Drawing it relies on knowing the trig functions of 30°, 45°, and 60° from two other shapes. For 45°, draw a square with sides of length 2. Then a diagonal will bisect two opposite angles into 45°. From the Pythagorean theorem, that diagonal will have length $2\sqrt{2}$ and thus half of the bisected square becomes the following isoceles triangle.

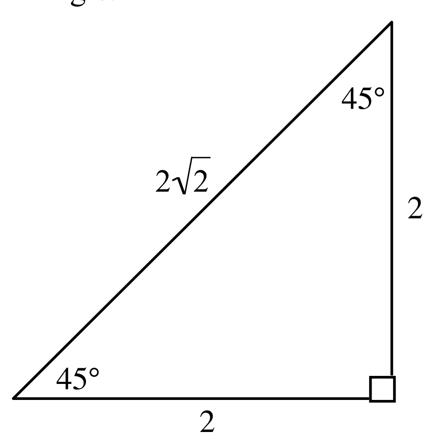

Figure 1

From this diagram, we read off

$$\cos 45° = \boxed{\frac{1}{\sqrt{2}}} = \sin 45° \tag{6}$$

and

$$\tan 45° = \boxed{1}\,. \tag{7}$$

(Take reciprocals to get the other three trig functions for this and the other shapes below.) The next shape is an equilateral triangle with sides of length 2. Then a bisector through one angle will divide the equilateral triangle into a right triangle with angles of 30° and 60° as sketched in Fig. 2, where the length of the vertical arm is obtained from the Pythagorean theorem.

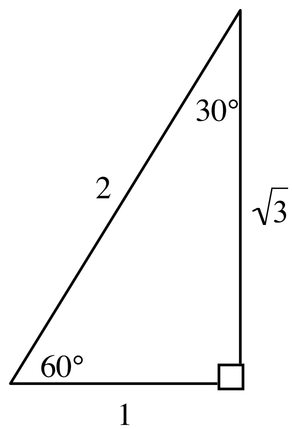

Figure 2

From this diagram, we read off

$$\cos 60° = \boxed{\frac{1}{2}} = \sin 30°\,, \tag{8}$$

and

$$\cos 30° = \boxed{\frac{\sqrt{3}}{2}} = \sin 60°\,, \tag{9}$$

and

$$\tan 60° = \boxed{\sqrt{3}} = \frac{1}{\tan 30°}\,. \tag{10}$$

Finally, rotate the triangle in Fig. 1 clockwise by 30°. Then place the triangle in Fig. 2 along its southeast side. Also rotate a copy of Fig. 2 clockwise by 90° and fit it into the southwest corner of the diagram. Finally complete the rectangle by drawing in a 15–75–90 triangle along its northwest corner. Label the lengths of the sides of the latter triangle (by referencing the lengths of the two 30–60–90 triangles) to obtain Fig. 3 and read off from it the results already given in Eqs. (4) and (5), duplicating the results of the first method.

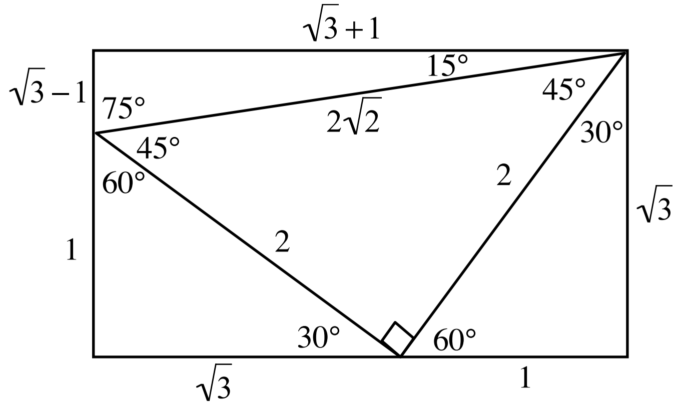

Figure 3

With these results in hand, one can freely use angles of 15° and 75° in problems without needing a calculator to evaluate their trig functions.