Speed and Amplitude of a Tsunami—C.E. Mungan, Summer 2006

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We treat a tsunami as a sinusoidal wave with period T whose wavelength λ is much larger than the ocean depth D. Consider a single crest, whose height h is equal to the amplitude of the tsunami and whose length is half a wavelength. A tube of water at the base of this crest with cross-sectional area A therefore has mass $m = \rho A \lambda / 2$ and is pushed forward by the water gauge pressure $p = \rho g h$, where ρ is the density of water. The water molecules composing this tube are accelerated from an initial speed of zero up to some horizontal particle speed v_H during the time $T/2$ that the crest hits it. Consequently Newton's second law can be written as

$$F = ma \quad \Rightarrow \quad \rho g h A = \rho A \lambda / 2 \frac{v_H}{T/2} \quad \Rightarrow \quad \boxed{v_H = \frac{ghT}{\lambda}}. \tag{1}$$

This is the horizontal speed of water molecules in the crest. At the same time, water molecules are also driven upward to some vertical speed v_V. Specifically, molecules move vertically a distance on the order of the amplitude in each period, so that

$$\boxed{v_V = \frac{h}{T}}. \tag{2}$$

Now consider a block of water that is one wavelength long in the ocean before the tsunami arrives. It has a lateral area of λD. But once the tsunami arrives, that same quantity of water becomes a pulse of height $D + h$ and length $\lambda - \Delta x$. By continuity (conservation of mass), the lateral area must remain unchanged,

$$\lambda D = (\lambda - \Delta x)(D + h) \quad \Rightarrow \quad \frac{\Delta x}{h} \cong \frac{\lambda}{D}. \tag{3}$$

A water molecule at the top leading corner of this block moves upward a distance h in time T with speed v_V, and inward a distance Δx in time T with speed $v_H = \Delta x / T$, so that

$$\frac{v_H}{v_V} = \frac{\Delta x}{h} = \frac{\lambda}{D} \quad \Rightarrow \quad T = \frac{\lambda}{\sqrt{gD}} \tag{4}$$

using Eqs. (1) to (3). Therefore the phase speed of the tsunami wave is

$$\boxed{c = \frac{\lambda}{T} = \sqrt{gD}}, \tag{5}$$

which establishes the well-known result that shallow water waves move with a speed proportional to the square root of the water depth. In a more exact treatment, this follows from

the small k limit of the dispersion relation $\omega^2 = gk\tanh(kD)$, where $\omega = 2\pi/T$ is the angular frequency and $k = 2\pi/\lambda$ is the angular wavenumber.

It is easy to see that as the tsunami approaches shore, it piles up into a narrower, taller pulse. The narrowing follows from the fact that $\lambda \propto D^{1/2}$. This occurs because $\lambda f \propto D^{1/2}$ according to Eq. (5) and the wave frequency f is a constant as it travels into water of different depths (just as a light wave has constant frequency but variable speed and wavelength as it travels from one medium into another). On the other hand, the power of a wave is proportional to its amplitude squared times its speed, $h^2 c$. (This follows from the fact that the particles oscillate vertically with an energy proportional to the amplitude squared while the wave propagates horizontally with speed c. See for example Giancoli Eq. 15-6.) Neglecting dissipative losses, we conclude that $h \propto c^{-1/2} \propto D^{-1/4}$ so that the wave gets taller as it approaches shore. The piling up is even more dramatic and becomes nonlinear when we account for viscous interactions with the beach bottom and resonances with coastal basins.