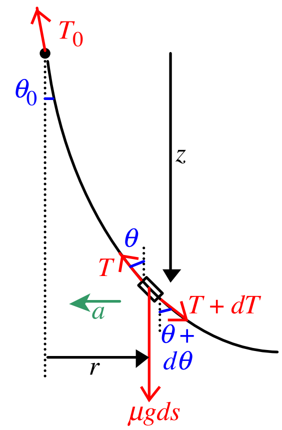
The Twirling Rope—C.E. Mungan, Fall 2014

A uniform rope of linear mass density μ is hanging from a single point at the origin and is then set into rotation about that point at angular speed ω. Find the equilibrium shape of the string after any initial transients have died away.



The diagram indicates the cylindrical coordinates (in black), the forces (in red), the angles of the string relative to the vertical (in blue), and the acceleration (in green) of two points along the string. At the topmost point, the tension is T_0 and the angle is θ_0. That angle cannot be zero because otherwise there would be no centripetal force on the rotating string as a whole. We will return to the two endpoints of the string later.

Consider the indicated small segment of the rope of length ds. Since the tension must decrease along the length of the string, $dT < 0$ in the diagram. On the other hand, assuming the string's radial position monotonically increases with position s along it, then $d\theta > 0$. Newton's second law in the radial direction predicts

$$T\sin\theta - (T + dT)\sin(\theta + d\theta) = \mu\omega^2 r ds \,, \tag{1}$$

while in the axial direction one obtains

$$T\cos\theta - (T + dT)\cos(\theta + d\theta) - \mu g ds = 0 \,. \tag{2}$$

The double-angle formulae for $d\theta \to 0$ predict

$$\sin(\theta + d\theta) = \sin\theta\cos d\theta + \cos\theta\sin d\theta = \sin\theta + \cos\theta d\theta \tag{3}$$

and

$$\cos(\theta + d\theta) = \cos\theta\cos d\theta - \sin\theta\sin d\theta = \cos\theta - \sin\theta d\theta \,. \tag{4}$$

Substituting Eqs. (3) and (4) into (1) and (2), one finds

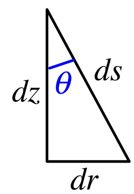
$$-dT\sin\theta - T\cos\theta d\theta = \mu\omega^2 r ds \tag{5}$$

and

$$-dT\cos\theta + T\sin\theta d\theta = \mu g ds \tag{6}$$

after dropping second-order differentials. Next, use the following triangle to deduce that

$$ds = \csc\theta\, dr\ . \tag{7}$$



Substitute Eq. (7) into (5) and (6), divide those two equations through by $d\theta$, and then simultaneously solve them for T and its derivative to obtain

$$T = \mu\left(g - \omega^2 r\cot\theta\right)\frac{dr}{d\theta} \tag{8}$$

and

$$\frac{dT}{d\theta} = -\mu\left(g\cot\theta + \omega^2 r\right)\frac{dr}{d\theta}\ . \tag{9}$$

If we now take the angular derivative of Eq. (8) and equate it to (9), we eliminate the tension to end up with a differential equation for the string's shape,

$$\left(g\tan\theta - \omega^2 r\right)\frac{d^2 r}{d\theta^2} + \left[g + \omega^2 r\left(\tan\theta + \sec\theta\csc\theta\right)\right]\frac{dr}{d\theta} = \left(\omega\frac{dr}{d\theta}\right)^2\ . \tag{10}$$

The equation can be recast in dimensionless form by defining the normalized centripetal acceleration $\alpha \equiv \omega^2 L/g$ where L is the length of the string, and the normalized radial coordinate $x \equiv r / L$. To numerically solve it, define the slope (effective speed) $\upsilon \equiv dx / d\theta$ and curvature (effective acceleration) $a = d^2 x/d\theta^2$ so that Eq. (10) becomes

$$a = \frac{\alpha\upsilon - 1 - \alpha x\left(\tan\theta + \sec\theta\csc\theta\right)}{\tan\theta - \alpha x}\upsilon\ . \tag{11}$$

Consider the initial conditions. The top end of the rope has coordinates $x_0 = 0$ and some starting angle θ_0. The vertical force balance on the entire string implies

$$T_0 = \mu L g\sec\theta_0\ . \tag{12}$$

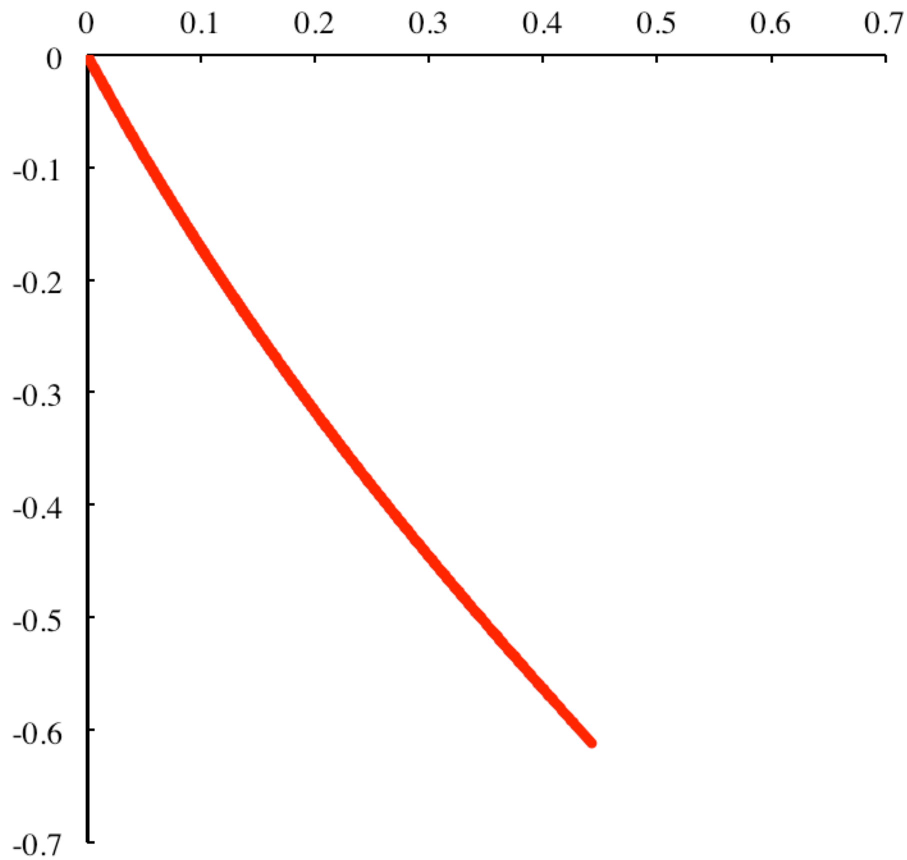
Set that equal to Eq. (8) evaluated at the top point to get $\upsilon_0 = \sec\theta_0$. We now have all the values we need to compute a_0 from Eq. (11) for any given α of interest. If we iterate in angular steps $\Delta\theta$, then the Euler-Cromer method gives the updated values of x and υ as

$$\upsilon = \upsilon_0 + a_0 \Delta\theta \quad \text{and} \quad x = x_0 + \upsilon \Delta\theta . \tag{13}$$

Combined with $\theta = \theta_0 + \Delta\theta$, we can now compute an updated value of a from Eq. (11). Furthermore, the vertical coordinate can be calculated from $dz = dr \cot\theta$ according to the preceding triangle diagram. Thus, if we define the normalized vertical coordinate $y \equiv z / L$ then

$$y = y_0 - \upsilon \cot\theta \Delta\theta \tag{14}$$

where the minus sign was inserted because the string extends downward. The string starts at $y_0 = 0$. The iteration is repeated until the end of the string is reached. That free end has $T = 0$ which according to Eq. (8) implies $\alpha x = \tan\theta$, i.e., we stop the iterations just before Eq. (11) diverges. For example, here is a resulting graph of the rope for $\alpha = 2$ and $\theta_0 = 28°$.



Interestingly, the rope can double back on itself and cross the vertical axis for large enough values of α according to EJP **19**, 379 (1998).