The Problem of Two Aces—C.E. Mungan, Summer 2010

Boas (3rd ed) problem 15.4.8 (in greatly paraphrased form) asks the following. You are dealt two cards face down from a shuffled deck. (i) What is the probability you got two aces? (ii) A computer does the dealing and checks the cards before laying them on the table. If neither card is an ace, it replaces both cards in the deck, reshuffles the deck, and repeats the process. As soon as the hand contains at least one ace, the computer lays the hand face down on the table. Now what is the probability you got two aces? (iii) Same as the second case, but this time the computer makes sure your hand contains the ace of spades before laying it down on the table.

Let us use the fact that the probability of getting what you want is the ratio of the number of ways of getting what you want to the number of ways of getting anything possible. Let us call the number of ways of getting what you want N_{win}. Then $N_{win} + N_{lose}$ is the number of ways of getting anything possible, where N_{lose} is the number of ways of getting what you do not want.

Note that the solutions must be the same regardless of whether or not you order the cards. (You could order them either temporally, by keeping track of which card was dealt first and which second, or by distinguishing the spatial positions of the two cards dealt.) If you order them, N_{win} and N_{lose} will both be twice as big as if you do not, because the number of ways of permuting two cards is[1] $P(2,2) = 2! = 2$. But when we take the ratio, this factor of two will cancel from the numerator and denominator. For simplicity, I will initially restrict my answers to the unordered case of combinations. But I will later explicitly consider order to get an alternate solution for the tricky case (ii).

So on to the answers. For convenience, let's denote the various aces by the following symbols: C = ace of clubs, D = ace of diamonds, H = ace of hearts, and S = ace of spades. In part (i), N_{win} is simply[1] $C(4,2) = 6$, because there are 6 ways of getting two aces, namely {CD,CH,CS,DH,DS,HS}. On the other hand, $N_{lose} = C(48,2) + C(4,1)C(48,1) = 1320$ since $C(48,2)$ is the number of ways of getting two non-aces and $C(4,1)C(48,1)$ is the number of ways of getting one ace and one non-ace (where we multiply the two, using the fundamental principle of counting). Thus, the probability of any two aces is $6/(6+1320) = 1/221$. You can also get this answer more directly by simply multiplying the probability of getting one ace (4/52) by the probability of getting a second ace (3/51).

In part (ii), there are still 6 ways of getting any two aces. But this time, N_{lose} is only $C(4,1)C(48,1) = 192$ since the possibility of getting two non-aces is excluded. Thus, the probability of receiving any two aces given one has risen to $6/(6+192) = 1/33$.

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[1]Here I use $P(N,n)$ and $C(N,n)$ to denote number of permutations and combinations of N objects taken n at a time.

Finally, in part (iii), N_{win} is simply $C(3,1) = 3$, because if we already have the ace of spades, there are only 3 ways of getting another ace, namely {CS,DS,HS}. By the same reasoning, $N_{lose} = C(48,1) = 48$ since there are 48 ways of getting a non-ace instead. Thus, the probability of two aces given the ace of spades is $3/(3+48) = 1/17$. Again, you can get this more simply by noting that if you have an ace of spades, the probability of getting any other ace is 3/51.

Since many people find it difficult to believe the answer to part (ii), here is another way to solve it, by this time distinguishing the order of the two cards. Consider two cards dealt from a deck of 52. Distinguish the cards' positions as L (left-hand card) and R (right-hand card). There are 51 distinct hands having S in the L position, 51 with H in the L position, 51 with C, and 51 with D, for a total of $4 \times 51 = 204$ allowable hands so far. Next consider how many additional hands are possible with an ace in the R position. There are 51 possible ways to get S in the R position, except that we must subtract 3 that have already been counted (namely HS, CS, and DS), for a total of 48 new hands. Likewise, there are 48 previously uncounted hands with H, 48 with C, and 48 with D in the R position, for a total of $4 \times 48 = 192$ additional legal hands. Thus, there are a grand total of 396 ordered hands that contain at least one ace. Of these hands, exactly 12 have aces in both the L and R positions. The odds that both cards are aces, given that at least one is an ace, is therefore $12 / 396 = 1 / 33$, the same solution as found before.

As one might have expected, the more information you have, the more your odds go up. If you carefully examine the possibilities, you will realize that there are twice as many ways to get any two aces as to get the ace of spades plus any other ace. But there are four times as many ways to get any ace plus any non-ace as to get the ace of spades plus any non-ace. Thus, your odds roughly get halved if you only know you have an ace but don't know its suit.

Thanks to Eric Nelson and Ann Reagan for some of these ideas. John Mallinckrodt tested these solutions by writing a program that deals hands according to these rules and counts the fraction of hands that contain two aces. John Denker emphasized that the dealing must be neutral—for example, it must be clear in case (ii) how the hands were prepared so that it is known they contain at least one ace. The solution to that case would be different (as emphasized by John Mourning) if the computer first separated out the aces, drew one out at random, then replaced the remaining 3 aces in the deck, reshuffled, and drew a second card out of the deck and dealt you those two drawn cards. Finally Philip Keller suggested comparing this problem to the following puzzle. "The Greens have two kids: what is the probability that they are both boys? The Whites have two kids, at least one of which is a boy: what is the probability that they are both boys? The Browns have two kids, the older one of which is a boy: what is the probability that they are both boys?" The answers are again different in all 3 cases (1/4, 1/3, and 1/2 respectively).