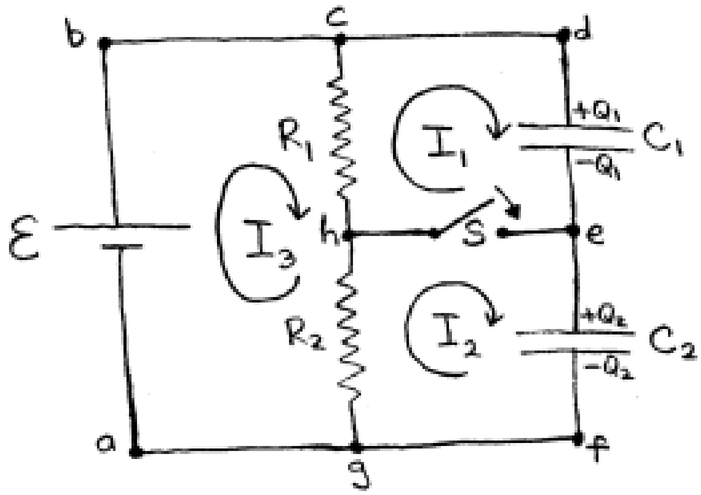
Time Constant for Charging a Pair of Capacitors—C.E. Mungan, Spring 2002

What is the time constant for re-establishing steady state after closing the switch S at t = 0?



Apply Kirchhoff's Voltage Loop (KVL) rule to loop hcde to get

$$-\dot{Q}_1 R_1 + I_3 R_1 - Q_1 / C_1 = 0 \tag{1}$$

where

$$\dot{Q}_1 \equiv dQ_1 / dt = I_1. \tag{2}$$

Similarly, loop abcg gives rise to

$$\varepsilon - I_3 R_1 + \dot{Q}_1 R_1 - I_3 R_2 + \dot{Q}_2 R_2 = 0. \tag{3}$$

Solve this for I_3 and substitute it into Eq. (1) to obtain

$$\frac{\varepsilon}{R_2} - \dot{Q}_1 + \dot{Q}_2 - \frac{R_1 + R_2}{R_1 R_2 C_1} Q_1 = 0 \tag{4}$$

In order to decouple Q_1 and Q_2, apply KVL to loop abdf so that

$$\varepsilon - \frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0 \quad \Rightarrow \quad \dot{Q}_2 = -\frac{C_2}{C_1} \dot{Q}_1 \tag{5}$$

where the second equality follows from taking the time derivative of the first equality. Substituting this into Eq. (4) gives

$$\varepsilon - \frac{C_1 + C_2}{C_1 / R_2} \dot{Q}_1 - \frac{R_1 + R_2}{C_1 R_1} Q_1 = 0. \tag{6}$$

If we compare this to the standard charging equation for a series RC circuit,

$$\varepsilon - R\dot{Q} - \frac{1}{C} Q = 0, \tag{7}$$

we can immediately read off the time constant from the coefficients of Eq. (6),

$$\boxed{\tau = RC = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} \left(C_1 + C_2\right)}. \tag{8}$$

Somewhat surprisingly, the time constant is identical to that of a series *RC* circuit consisting of two resistors R_1 and R_2 in parallel, and two capacitors C_1 and C_2 in parallel, i.e., the circuit obtained by interchanging C_1 and R_2 in our circuit.

It is informative to write down the general solution for the charges on the capacitors. Just before the switch is closed, the capacitors are in series so that

$$Q_{10} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} \varepsilon. \tag{9}$$

On the other hand, at long times after the switch is closed, the currents I_1 and I_2 fall to zero and the voltages across the capacitors equal those across the corresponding resistors which act like a voltage divider, so that

$$Q_{1\infty} = C_1 \frac{R_1}{R_1 + R_2} \varepsilon. \tag{10}$$

Thus, the equation describing the time dependence of Q_1 is

$$Q_1(t) = Q_{1\infty} - (Q_{1\infty} - Q_{10})e^{-t/\tau}. \tag{11}$$

This differs from the standard *RC* series circuit formula only in that our circuit *cannot* start out with initially uncharged capacitors. In the distant past when you connected the battery across the capacitors, they charged up nearly instantaneously to their initial values, with some small time constant (determined by the internal resistance of the wires and battery) unrelated to Eq. (8).

From the symmetry of the circuit, we can obtain all of the analogous equations in this document for the second capacitor by interchanging the subscripts “1” and “2” in all of the preceding equations. Note in particular that this leaves Eqs. (8) and (9) unchanged.