Mass on a Vertical Spring—C.E. Mungan, Fall 1999

The following is not particularly profound, but is a subtle point often glossed over in introductory textbooks. I remember it caused me some confusion as an undergraduate.

Introductory treatments of simple harmonic motion (SHM) generally draw and discuss pictures of an ideal (i.e., massless, Hookean, and internally frictionless) horizontal spring, one end of which is attached to a wall and to the other end of which a point mass is attached. External dissipation such as friction with the floor, air resistance, and the like are neglected until damping is discussed. Using this picture, concepts such as Hooke's law and elastic potential energy are appropriately developed in the context of forces and mechanical energy, respectively. The confusion enters in when the instructor proceeds to demonstrate this and solve problems with a mass hanging vertically from a spring whose other end is attached to a horizontal post. "Why doesn't the gravitational force and the gravitational potential energy screw up the forces and the mechanical energy?" an intelligent student might reasonably ask.

In point of fact, it is a simple exercise to verify that the following two statements which are usually proven only for the horizontal case also hold for the vertical case, so that gravity can be ignored entirely:

(i) a mass oscillates on a spring with SHM about its equilibrium position;
(ii) there is a continuous exchange of energy between kinetic and potential energies, where at the equilibrium position one has KE_{max} and $PE = 0$, while at the endpoints one has $KE = 0$ and PE_{max}, where $PE = \frac{1}{2}kx^2$ with x being the displacement from equilibrium.

The key point is that the equilibrium position is defined as the position of the mass where it freely remains stationary. This is identical to the unstrained position of the spring (i.e., the length it has with no mass attached to it) only in the horizontal case; in the vertical case, the equilibrium position lies a distance d below the unstrained position, where the force balance implies $d = mg/k$. (This is the only effect of gravity and is relevant only for rather artificial textbook problems where the unstrained length of the spring is specified rather than the equilibrium length.) The proof of the above two statements is relegated to the appendix.

Let me cite a few examples of problems which can be solved using these ideas.

1. Example 12 on page 293 of Cutnell & Johnson: A 200-g ball is attached to the end of a vertical spring of 28-N/m stiffness constant. The ball is initially supported so that the spring is unstrained. If it is suddenly dropped, what is its maximum displacement h from the point of release?

Evidently the points of release and of maximum extension are the two turning points of the SHM. Hence the equilibrium position must lie midway in-between . Therefore $h = 2d = 2mg/k$. Also see problem 73; unfortunately the Solutions Manual foolishly does not point out the logic developed here.

2. Problem 53 on page 304 of Cutnell & Johnson: A 2-kg ball is hanging from the end of a vertical spring of 50-N/m stiffness constant. The ball is pulled downward by 20 cm and released from rest. Compute the kinetic energies at $h = 0$, 20, and 40 cm above the point of release.

According to the above way of thinking, it is not useful to attempt to disentangle gravitational and elastic potential energy. So I would obtain the answers from

$$PE = \tfrac{1}{2}k(20 \text{ cm} - h)^2 \text{ and } KE = E - PE \text{ where } E = \tfrac{1}{2}k(20 \text{ cm})^2 = 1 \text{ J}$$

where the ball's mass is thus not needed. The KE is therefore 0, 1, and 0 J at the respective three points of interest. This agrees with the Solutions Manual, as obtained with much more effort. The solutions for the PE of course do not agree but I would argue those have no physical meaning anyhow. For example, the problem shamefully omits to specify the zero level for the gravitational PE!

3. Perhaps you now think that while what I have done is fine, it is simply not as systematic and complete as separately treating gravitational and elastic potential energy. If so, I invite you to throw the following Mazur question at your students: An object hangs motionless from a spring. When the object is pulled down, does the sum of the elastic and gravitational potential energies of the object increase, decrease, or remain constant?

Again, according to the above way of thinking, the total PE increases whenever you displace a spring from its equilibrium point, because when you release it the object will by definition be pulled back toward that point. No need to fuss over the fact that the gravitational PE goes down and the elastic PE goes up!

Appendix

Various positions of the mass on a vertical spring can be sketched as follows,
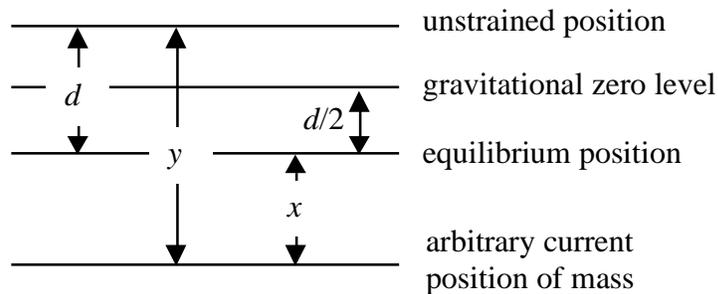
where for convenience I have positioned the zero level of the gravitational PE midway between the unstrained and equilibrium positions. (It is important to notice that the gravitational zero level is arbitrary since gravitational PE is linear in the height, but elastic PE is not linear in the displacement and thus the elastic zero level is necessarily at the unstrained position if you wish to use the standard formula.) As already proven above, $d = mg/k$.

Proof of statement (i): Newton's second law can be written as
$$-\omega^2 y + g = \ddot{y}$$
with downward chosen to be positive. This has solution
$$y = A\cos(\omega t + \phi) + d.$$
The proof follows by observing that $x \equiv y - d$.

Proof of statement (ii): The elastic and gravitational potential energies are defined with respect to their zero levels, so that
$$PE = \tfrac{1}{2}k(x+d)^2 - mg(x + \tfrac{1}{2}d).$$
Substituting for d and simplifying gives $PE = \tfrac{1}{2}kx^2$. Q.E.D.