Power to Create a Water Jet—C.E. Mungan, Fall 2003

A hose creates a column of water with radius r at ground level that shoots straight upward to a maximum height h. What power is required to create this jet? (This is based on problem 91 in chapter 8 of Giancoli.)

The water must come out of the hose with speed $\upsilon_0 = \sqrt{2gh}$ in order to reach height h. The time required for a water drop to travel from the base to the top of the water column is $T = \sqrt{2h/g}$. We can find the solution several different ways.

Solution A: A small bit of water of mass dm leaves the nozzle with kinetic energy (but no potential energy relative to the nozzle) in a time dt, so that

$$P = \frac{dE}{dt} = \frac{\frac{1}{2} dm\, \upsilon_0^2}{dt}. \tag{1}$$

Let the bit of water be in the shape of a pancake with radius r and height dy, so that

$$dm = \rho dV = \rho \pi r^2 dy = \rho \pi r^2 \upsilon_0 dt. \tag{2}$$

where $\rho = 1$ g/cm^3 is the density of water, since the pancake has cylindrical volume dV. Note that if we integrate both sides over the entire water column, we find its mass to be

$$M = \rho \pi r^2 \upsilon_0 T = 2\rho \pi r^2 h \tag{3}$$

where the second step followed by substituting for υ_0 and T from the top of this page. This is twice as large as one might suppose by naïvely integrating the third term in Eq. (2), because the water column slows down and hence widens as it rises, assuming the column always remains continuous and incompressible. (In particular, we see that the cross-sectional area of the column averaged over its height must be double its base area of πr^2. This can also be proven by noting that $A\upsilon$ is constant, where υ is the upward speed of a slice of the column with cross-sectional area A.) Alternatively one can derive Eq. (1) by noting that the bit of water reaches the top with gravitational potential energy but no kinetic energy, so that $dE = dm\,gh$. Substituting Eq. (2) into (1) together with $\upsilon_0 = \sqrt{2gh}$ now implies

$$\boxed{P = \sqrt{2}\rho \pi r^2 (gh)^{3/2}}. \tag{4}$$

Solution B: Noting that the power is constant and using the generalized form of Newton's second law, we have

$$P = P_{\text{ave}} = (\vec{\mathbf{F}} \cdot \vec{\boldsymbol{\upsilon}})_{\text{ave}} = \frac{dp}{dt}\upsilon_{\text{ave}} = \frac{dm \upsilon_0}{dt}\frac{\upsilon_0}{2} \tag{5}$$

assuming that the force uniformly accelerated the bit of water dm from zero speed up to the nozzle speed υ_0. Alternatively, note that the entire column is in steady state. Thus the force exerted must balance the weight of the column,

$$F = Mg = \frac{M\upsilon_0}{T} = \frac{dm}{dt}\upsilon_0 . \tag{6}$$

The last equality follows from comparison of Eqs. (2) and (3), and arises because water flows out of the nozzle at a constant rate. [This result implies that we can also interpret *dp*/*dt* in Eq. (5) as the loss in momentum of the bit of water over the height of the column, with $\upsilon_{\text{ave}} = \upsilon_0 / 2$ since the water is uniformly decelerated. That is, the work done on a parcel of water beginning from rest as it is pumped through the nozzle is equal and opposite to the work done by gravity on this water as it rises up the column back to rest.] Equation (5) reproduces Eq. (1) and hence leads to the same solution (4).