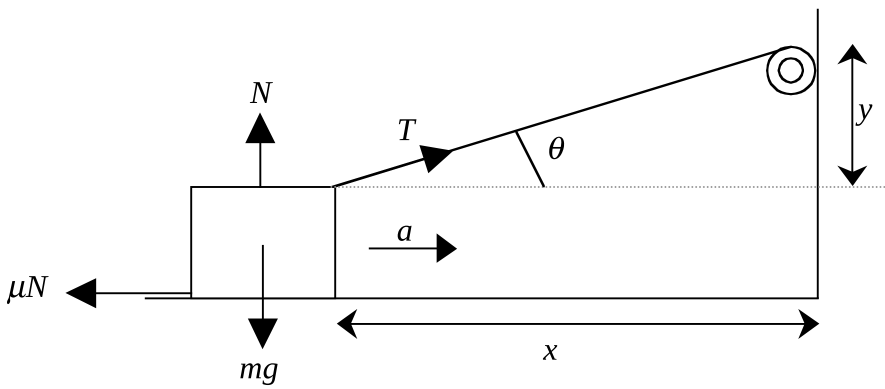
Box Pulled on a Rough Surface by a Winch—C.E. Mungan, Spring 2002

A winch of radius R at height y turning at a constant angular speed ω is pulling a box of mass m along a rough surface with coefficient of kinetic friction μ. Find the acceleration of the box and the tension in the cable at the instant the box is a horizontal distance x from the winch.



Assuming the cable does not slip on the winch, it is being reeled in at a constant speed $\upsilon = R\omega$. Denote the length which has not yet been reeled in as $L = (x^2 + y^2)^{1/2}$. Since υ is the rate at which this length decreases with time, the chain rule implies that

$$\upsilon = -\frac{dL}{dt} = \frac{x}{\sqrt{x^2 + y^2}}\upsilon_x \quad \Rightarrow \quad \upsilon_x = \frac{\upsilon L}{x} \tag{1}$$

where $\upsilon_x = -dx/dt$ is the horizontal speed of the box. (This result says that $\upsilon_x = \upsilon / \cos\theta$. Only the component of the box's velocity directed toward the winch contributes to a change in the cable's length.) Differentiating this in turn gives us the acceleration of the box,

$$a = \frac{d\upsilon_x}{dt} = \frac{\upsilon^2 y^2}{x^3} \tag{2}$$

after a little bit of algebra and care with signs. Notice that the box must be gaining speed regardless of the values of θ and μ.

We next apply Newton's second law both vertically,

$$N = mg - T\sin\theta, \tag{3}$$

and horizontally,

$$T\cos\theta - \mu N = ma. \tag{4}$$

Substituting Eqs. (2) and (3) into (4) and relating the trignometric functions to the distances labeled on the diagram gives the somewhat unwieldy expression,

$$T = m\left(\mu g + \frac{\upsilon^2 y^2}{x^3}\right)\frac{\sqrt{x^2 + y^2}}{x + \mu y}. \tag{5}$$

I have graphed the box's speed and acceleration, as well as the tension in the cable, at the top of the next page using m = 1 kg, μ = 0.5, g = 9.8 m/s^2, y = 1 m, and υ = 1 m/s. It is no great surprise to find that in the limit as $x \to \infty$ we have $\upsilon_x = \upsilon$, $a = 0$, and $T = \mu mg$, and that all three quantities diverge as $x \to 0$.