

## Transformation Equation for Center-of-Mass Work—C.E. Mungan, Fall 2006

In this paper I derive an equation that relates classical center-of-mass work  $W$  performed in an inertial frame to the work  $W'$  done in some other frame which in general is noninertial because its origin translationally accelerates with respect to the origin of the inertial frame. Specifically, let its instantaneous velocity be  $\vec{V}$  and its acceleration be  $\vec{A}$ .

Denote quantities measured in the noninertial frame by primes and corresponding quantities in the inertial frame by an absence of primes. Then the Galilean velocity transformation is

$$\vec{v} = \vec{v}' + \vec{V} . \quad (1)$$

If we multiply every term in this expression by  $dt$ , Eq. (1) can be rewritten as

$$d\vec{r} = d\vec{r}' + \vec{V}dt , \quad (2)$$

describing the transformation between displacements. On the other hand, if  $\vec{v}$  is the velocity of a particle or of the center of mass of a system of interest whose total mass is  $m$ , then Eq. (1) can alternatively be written as

$$\vec{p} = \vec{p}' + m\vec{V} \quad (3)$$

which describes the transformation between linear momenta. Finally, taking the time derivative of each term in Eq. (1) gives the acceleration transformation,

$$\vec{a} = \vec{a}' + \vec{A} , \quad (4)$$

and multiplying each term by  $m$  results in

$$\vec{F} = \vec{F}' + m\vec{A} , \quad (5)$$

describing the transformation between forces. Since the unprimed frame is inertial,  $\vec{F}$  is the net *real* force on the system, while to preserve the form of Newton's second law in the noninertial frame,  $\vec{F}'$  must be equal to the sum of the real forces and the translational pseudo force  $\vec{F}_{\text{pseudo}} = -m\vec{A}$ .

Center-of-mass work is defined as the line integral of the net force over the displacement of the system's center of mass,

$$\begin{aligned} W &\equiv \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot (d\vec{r}' + \vec{V}dt) = \int (\vec{F}' + m\vec{A}) \cdot d\vec{r}' + \int \vec{F} \cdot \vec{V}dt \\ &= \int \vec{F}' \cdot d\vec{r}' + m \int \frac{d\vec{V}}{dt} \cdot \vec{v}'dt + \int \vec{V} \cdot d\vec{p} \\ &= W' + \int \vec{p}' \cdot d\vec{V} + \int \vec{V} \cdot d\vec{p} = W' + \int (\vec{p} - m\vec{V}) \cdot d\vec{V} + \int \vec{V} \cdot d\vec{p} \\ &= W' + \int d(\vec{p} \cdot \vec{V}) - \frac{1}{2}m \int d(\vec{V} \cdot \vec{V}) \end{aligned} \quad (6)$$

using Eqs. (2), (3), and (5). The final expression can be written in a particularly compact way by noting that

$$\bar{\mathbf{p}}_{\text{ave}} \equiv \frac{1}{2}[\bar{\mathbf{p}} + \bar{\mathbf{p}}'] = \frac{1}{2}[\bar{\mathbf{p}} + (\bar{\mathbf{p}} - m\vec{\mathbf{V}})] = \bar{\mathbf{p}} - \frac{1}{2}m\vec{\mathbf{V}} \quad (7)$$

so that

$$\boxed{W = W' + \Delta(\bar{\mathbf{p}}_{\text{ave}} \cdot \vec{\mathbf{V}})} \quad (8)$$

is the desired transformation equation for work.

A faster derivation of this result assumes that the work-kinetic-energy theorem is valid in a noninertial frame,

$$\begin{aligned} W' &= \Delta K' \equiv \Delta\left[\frac{1}{2}m\vec{\mathbf{v}}' \cdot \vec{\mathbf{v}}'\right] = \Delta\left[\frac{1}{2}m(\vec{\mathbf{v}} - \vec{\mathbf{V}}) \cdot (\vec{\mathbf{v}} - \vec{\mathbf{V}})\right] \\ &= \Delta K - \Delta(\bar{\mathbf{p}} \cdot \vec{\mathbf{V}}) + \Delta\left(\frac{1}{2}mV^2\right) = W - \Delta(\bar{\mathbf{p}} \cdot \vec{\mathbf{V}} - \frac{1}{2}mV^2) \end{aligned} \quad (9)$$

because Eq. (7) converts the last expression into Eq. (8).

Let's consider a couple of simple applications of this result:

Case 1—Suppose  $\vec{\mathbf{V}} = \text{constant}$  so that the primed frame is inertial. Then Eq. (9) becomes

$$W' = W - \vec{\mathbf{V}} \cdot \Delta\bar{\mathbf{p}} = W - \vec{\mathbf{V}} \cdot \vec{\mathbf{I}} \quad (10)$$

where the impulse  $\vec{\mathbf{I}} = \int \vec{\mathbf{F}} \cdot dt = \int \vec{\mathbf{F}}' \cdot dt$  is frame invariant since  $\vec{\mathbf{A}} = 0$  in Eq. (5). In contrast, the work is only frame invariant if the impulse (or at least its component in the direction of the relative velocity of the frames) is zero. For example, if I drop a stone from rest, the gravitational work that the earth does on the stone can be positive, negative, or zero depending on the vertically moving inertial frame that I chose to view the process from; however, the sum of the net positive work done on both the earth and stone is frame invariant (and equal to the loss in gravitational potential energy of this isolated system), as I discuss in JWAS **92**, 7 (Summer 2006). Equation (10) is equivalent to Eq. (29) in AJP **60**, 356 (1992).

Case 2—Suppose that we choose to view things from the center-of-mass frame  $\vec{\mathbf{V}} = \bar{\mathbf{p}}/m$  so that  $\bar{\mathbf{p}}' = 0$ . Then  $\bar{\mathbf{p}}_{\text{ave}} = \bar{\mathbf{p}}/2$  so that Eq. (8) becomes

$$W' = W - \Delta\left(\frac{\bar{\mathbf{p}} \cdot \bar{\mathbf{p}}}{2m}\right) = W - \Delta K = 0 \quad (11)$$

which simply says that no center-of-mass work is done in the frame in which the center of mass is at rest.

Note that  $W' = \int \vec{\mathbf{F}}' \cdot d\vec{\mathbf{r}}'$  is the sum of the work done in the primed frame by both real and pseudo forces, as discussed following Eq. (5). (Do not confuse work done by a pseudo force with pseudo work, which is a synonym for center-of-mass work!) It might be interesting to generalize Eq. (8) so that it includes not just translational pseudo forces, but also rotational pseudo forces (e.g., centrifugal and Coriolis forces) by replacing Eq. (1) with  $\vec{\mathbf{v}} = \vec{\mathbf{v}}' + \vec{\mathbf{V}} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}'$  where  $\vec{\boldsymbol{\omega}}$  describes the rotation of the primed frame relative to the unprimed frame.