Problem 1
A cyclist rides along a long straight road. For the first 5.0 miles, the cyclist warms up at a pace of 16 mph. For the second 5.0 miles the cyclist time trials at a pace of 26 mph.
In the following assume the cyclist gets quickly enough up to speed such that you can approximately neglect the transition time between the warm-up and time-trial segments.

a) What is the average velocity of the cyclist over the full 10 mile distance?
b) If we now assume that the warm up is 5.0 miles at 10 mph, after which the cyclist turns around and returns to the starting point at 35 mph. On this trip what is the cyclist’s average velocity? average speed?

GUIDANCE: Follow carefully the definition of average velocity. The correct answer may not be the one that you intuitively expect. For the answer to become intuitive, you might need to reflect a bit on the process of averaging in general and time averaging in this particular instance. Did you need to know the distance traveled by the cyclist to find the average velocity and speed in part a) or b)?

Solution
a) The definition of average velocity is the total displacement over the time taken for that displacement

\[ v_{avg} = \frac{\Delta x_{total}}{\Delta t_{total}} \]

since the cyclist is traveling in the same direction for the whole trip, the total displacement is \( \Delta x_{total} = 5\text{mi} + 5\text{mi} = 10\text{mi} \). To find the total time taken for the trip we must find the time for each part

\[
\begin{align*}
\frac{\Delta x_1}{\Delta t_1} &= v_1 \rightarrow \Delta t_1 &= v_1 &= \frac{5\text{mi}}{16\text{mph}} = \frac{5}{16}\text{hrs} \\
\frac{\Delta x_2}{\Delta t_2} &= v_2 \rightarrow \Delta t_2 &= v_2 &= \frac{5\text{mi}}{26\text{mph}} = \frac{5}{26}\text{hrs} \\
\Delta t_{total} &= \Delta t_1 + \Delta t_1 &= \frac{5}{16}\text{hrs} + \frac{5}{26}\text{hrs} = \frac{105}{208}\text{hrs}
\end{align*}
\]

the average velocity is then

\[ v_{avg} = \frac{10\text{mi}}{\frac{105}{208}\text{hrs}} = \frac{2080}{105}\text{mph} = 19.8\text{mph} \]

b) For this trip the cyclist starts and ends at the same spot this means that the total displacement is \( \Delta x = x_f - x_i = 0\text{mi} \). This tells us that the average velocity is must also be zero \( v_{avg} = \frac{\Delta x}{\Delta t} = 0\text{mi} = 0\text{mph} \).

The average speed is a different matter. The total distance traveled is still 10 miles as in (a). The time for each section is found in the same way as before

\[
\begin{align*}
\frac{\Delta x_1}{\Delta t_1} &= v_1 \rightarrow \Delta t_1 &= v_1 &= \frac{5\text{mi}}{10\text{mph}} = \frac{1}{2}\text{hrs} \\
\frac{\Delta x_2}{\Delta t_2} &= v_2 \rightarrow \Delta t_2 &= v_2 &= \frac{5\text{mi}}{35\text{mph}} = \frac{1}{7}\text{hrs} \\
\Delta t_{total} &= \Delta t_1 + \Delta t_1 &= \frac{1}{2}\text{hrs} + \frac{1}{7}\text{hrs} = \frac{9}{14}\text{hrs}
\end{align*}
\]

giving and average of

\[ s_{avg} = \frac{\text{total distance}}{\Delta t_{total}} = \frac{10\text{mi}}{\frac{9}{14}\text{hrs}} = \frac{140}{9}\text{mph} = 15.6\text{mph} \]
Interestingly we did not have to know the distance here. If we just knew each segment of the trip was the same distance $D$ we could follow everything above algebraically.

\[
\begin{align*}
    v_{\text{avg}} &= \frac{\Delta x_{\text{total}}}{\Delta t_{\text{total}}} \\
                &= \frac{\Delta x_1 + \Delta x_2}{\Delta t_1 + \Delta t_2} \\
                &= \frac{D + D}{\frac{D}{v_1} + \frac{D}{v_2}} \\
                &= \frac{2D}{\frac{v_1 v_2}{v_1 + v_2}} = 2 \frac{v_1 + v_2}{v_1 v_2}
\end{align*}
\]

and as you can see the $D$ cancels out.
Problem 2
The position as a function of time for a car shooting for a land speed record is modeled approximately by
the following formula,
\[ x(t) = 135[t + 15(e^{-t/15} - 1)]. \]
It is understood that \( x \) is in meters and \( t \) is in seconds. What is the car’s
a) velocity at \( t = 15, 30, \) and \( 45 \) sec? (answers will be in m/s with the implied units)
b) velocity very very far in the future if this formula were valid for very large \( t \)?

Solution
a) We can simply find the velocity at any time by taking the derivative of the position with respect to time
\[
v = \frac{dx}{dt} = \frac{d}{dt} 135[t + 15(e^{-t/15} - 1)]
= 135[\frac{d}{dt}t + 15\frac{d}{dt}(e^{-t/15} - 1)]
= 135 \left[ 1 - e^{-t/15} \right]
= 135 - 135e^{-t/15}
\]
plugging in the times gives
\[
\begin{align*}
v (15s) &= 135 - 135e^{-15/15} = 85.33 m/s \\
v (30s) &= 135 - 135e^{-30/15} = 116.7 m/s \\
v (45s) &= 135 - 135e^{-45/15} = 128.3 m/s
\end{align*}
\]
b) We could just start plugging in very large times and see what happens to our formula for velocity
above, but really what we want is the limit of this velocity as time goes to infinity (which is very far in the
future). As time gets large, the first 135 in the expression for \( v(t) \) doesn’t change, however as \( t \) gets very
large \( e^{-t/15} \) will start to get very small ( \( e \) to a very negative number is very small). This means for large
times the second term start to become very small, and eventually it won’t really contribute at all meaning
at very large times
\[
v (t \to \infty) = 135 m/s
\]
Problem 3
At \( t = 0 \), a particle is at rest at the origin. From \( t = 0 \) to \( t = 8.0 \) s, the particle experiences an acceleration of \( a(t) = 3(1 - t/8) \) where \( a \) has units of \( \text{m/s}^2 \) and \( t \) is in seconds. After \( t = 8 \) s, \( a = 0 \).

a) Create a nice graph of \( a(t) \) over this time interval. b) What is the particle’s velocity at \( t = 8 \) s? c) What was the particles average acceleration over the first 8 seconds? d) How would your answer to part b) change if the particle started with an initial velocity of \(-17 \) m/s at \( t = 0 \)s?

Solution

a)

![Graph of a(t) over time interval](image)

b) From the fundamental theorem of calculus we know that

\[
v(t_f) - v(t_i) = \int_{t_i}^{t_f} \frac{dv}{dt} dt
\]

but we know that \( a = \frac{dv}{dt} \) so that

\[
v(t_f) - v(t_i) = \int_{t_i}^{t_f} a(t) dt
\]

inserting \( t_i = 0 \) and \( t_f = 8s \) we could just do this integral. However, we can just look at the graph and remember that the integral is the area under the curve. For \( t_i = 0 \) and \( t_f = 8s \) that is just the area of the triangle so that

\[
v(t_f) - v(t_i) = \frac{1}{2} \times 3 \times \frac{3m}{s^2} \times 8sec
\]

\[
= 12 \frac{m}{s}
\]

we know that the initial velocity was \( 0 \)m/s so \( v(t_i) = 0 \) giving us

\[
v(8sec) = 12m/s
\]

c) The average acceleration is defined as

\[
a_{avg} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}
\]
plugging in what we found in b) gives

\[ a_{avg} = \frac{12\text{m/s} - 0\text{m/s}}{8\text{sec}} = \frac{3}{2}\text{m/s}^2 = 1.5\text{m/s}^2 \]

d) If the initial velocity was now given by \( v(t_i) = -17\text{m/s} \) looking above we can see that we only have to add this initial velocity to our answer for part b)

\[
v(t_f) = \left( \int_{t_i}^{t_f} a(t) \, dt \right) + v(t_i)
\]
\[
= 12\text{m/s} + (-17\text{m/s})
\]
\[
= -5\text{m/s}
\]