Introduction to Matrices and Determinants

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The Pauli matrices ***** TO BE DEVELOPED

Problems in circuit analysis, applications of Newton’s laws and problems such as the mini-max (pork sausage) problem in economics can be reduced to solving a set of linear algebraic equations.

$$
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1
$$
$$
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2
$$
$$
\ldots
$$
$$
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = c_n
$$

[MD.1]

that can be represented in a compact matrix form:

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Basic matrix definitions, properties and procedures are to be presented as background supporting the study of systems of linear algebraic equations.

A matrix is a rectangular array of values called elements. The values may be scalar values selected from a scalar field such as the real or complex numbers, scalar valued algebraic expressions or more complicated entities. This introduction is to focus on elements that evaluate to scalars.

* The elements of a matrix are scalars. Scalars are members of a field and they satisfy the field axioms. Use the axioms to justify operations on the scalar elements.

http://mathworld.wolfram.com/FieldAxioms.html

The symbol \( a_{ij} \) is the value in the \( i^{th} \) row and \( j^{th} \) column of the rectangular array, the matrix \( \mathbb{A} \).

\[
\mathbb{A} = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \cdots & \cdots & \cdots & \cdots \\
    a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

\( \text{an } m \times n \text{ (row x column) matrix} \)

**Notation Alert:** Here the elements of a matrix \( \mathbb{A} \) are represented as \( a_{ij} \). A common and perhaps superior representation of the elements of \( \mathbb{A} \) is \( A_{ij} \).

Addition is defined for matrices of the same dimensions \( m \times n \) as they are for vectors. The addition is component-wise. The sum matrix has elements that are the sum of the corresponding elements in the individual matrices.

\[
[\mathbb{A} + \mathbb{B}]_{ij} = a_{ij} + b_{ij}
\]

\[
\mathbb{A} + \mathbb{B} = \begin{bmatrix}
    a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{12} \\
    a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\
    \cdots & \cdots & \cdots & \cdots \\
    a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn}
\end{bmatrix}
\]

A matrix multiplied by a scalar \( k \) is the matrix with each element multiplied by that scalar. The scalar multiple has the same dimensions as the original matrix.
$$k \mathbf{A} = \begin{bmatrix} k \ a_{11} & k \ a_{12} & \ldots & k \ a_{1n} \\ k \ a_{21} & k \ a_{22} & \ldots & k \ a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k \ a_{m1} & k \ a_{m2} & \ldots & k \ a_{mn} \end{bmatrix}$$

The additive identity is $\mathbf{0}$ is the matrix in which every element is identically zero. It is also to be called the zero matrix or the **null matrix**. Note that there are distinct null matrices for each distinct set of dimensions $m \times n$.

The additive inverse is defined as $(-1) \mathbf{A}$, and subtraction is defined as the addition of the additive inverse.

$$-\mathbf{A} = (-1) \mathbf{A} = \begin{bmatrix} -a_{11} & -a_{12} & \ldots & -a_{1n} \\ -a_{21} & -a_{22} & \ldots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & \ldots & -a_{mn} \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \ldots & a_{1n} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} & \ldots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{n1} & a_{m2} - b_{n2} & \ldots & a_{mn} - b_{nn} \end{bmatrix}$$

**A Vector Space of Matrices**: *(Ignore this paragraph if you are not familiar with vector spaces.)*

Taken together, these properties qualify the set of all $m \times n$ matrices with elements $a_{ij}$ that are any scalar values from a field $\mathbb{F}$ as an $m \times n$ dimensional vector space. One basis set for the space is the set of all matrices that have elements $\delta_{is}\delta_{ij}$ for $1 \leq s \leq m; 1 \leq t \leq n$. That is: the collection of $m \times n$ matrices in which one element is 1 and all others are 0 expanded until each of the $m \times n$ element positions has its matrix with a 1 in its position plus 0's elsewhere.

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \quad \text{**The Kronecker Delta Notation**}$$

**Matrix Multiplication**: The sets of equations represented by a matrix can represent an operation or transformation such as a rotation by $15^\circ$ about the z-axis. It results that a sequence of operations or
transformations can be represented by a sequence of matrices acting. This action-in-sequence is equivalent to a matrix multiplication. The product of two matrices is defined by an equation for an element in the product matrix.

\[(A \cdot B)_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}\]  

[MD.3]

The \(ij\) element of the product involves the values from the \(i^{th}\) row of the leftmost matrix and the \(j^{th}\) column of the rightmost matrix. The sum over \(\ell\) requires that the number of columns in \(A\) be the same as the number of rows in \(B\). If not, the product \(A \cdot B\) is simply *not defined*. The result is a matrix with the same number of rows as \(A\) and the same number of columns as \(B\). Thus an \(m \times n\) matrix times an \(n \times p\) matrix yields an \(m \times p\) matrix. Also, the matrix product \(A \cdot B\) is not necessarily equal to \(B \cdot A\). That is: matrix multiplication is *not commutative*.

**Sample product calculation:**

\[
\begin{bmatrix}
  2 & 1 & 2 \\
  3 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
  -1 & 2 \\
  2 & 0 \\
  3 & 1
\end{bmatrix}
= \begin{bmatrix}
  2(-1)+1(2)+2(3) & 2(2)+0+2(1) \\
  3(-1)+0+2(3) & 3(2)+0+2(1)
\end{bmatrix}
= \begin{bmatrix}
  6 & 6 \\
  3 & 8
\end{bmatrix}
\]

The \(ij\) element of the product matrix can be computed as the dot product of a vector with the components in the \(i^{th}\) row of the first matrix with a vector that has the components in the \(j^{th}\) column of the second matrix. The informal procedure is the “hook’em-horns” method that guides the process as follows. To compute the \(ij\) element of the product, fold the middle two fingers of your right hand back to touch the palm leaving the index and little fingers extended. Point the tip of the index finger at the first element in row \(i\) of the leftmost matrix and the little finger at the last element of that same row. Visualize that row of values glued to your finger tips as you rotate them in space to direct the index finger at the top element in the column \(j\) of the second matrix and the little finger at the bottom element in that same column. Compute the products of elements at corresponding locations relative to your finger tips and add. While the “hook’em horns” display is repulsive (*to 60's Rice grads*), it is nonetheless helpful while multiplying matrices.

Matrix multiplication is associative; it is not commutative.

\[(A \cdot B) \cdot C = A \cdot (B \cdot C) \quad ; \quad A \cdot B \text{ not necessarily equal } B \cdot A\]

**Sample Calculation: Not Commutative**  
Consider the example:
\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\text{ compare to }
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

For this example, \( AB = -BA \). An example of non-commuting matrix multiplication has been displayed. \textit{Matrix multiplication is not commutative}. Matrix multiplication is associative. The associative property of matrix multiplication is to be established in a problem.

\textbf{Special Terms:}

\textbf{Square matrix:} A matrix with the same number of rows and columns. The product of two \( n \times n \) square matrices is another square matrix of the same size.

\textbf{The (main body) diagonal:} (used with square matrices) The descending diagonal of elements from the upper left to the lower right which includes all the elements with the same row and column numbers.

\textbf{Trace of a Matrix:} (defined for square matrices) The trace of a matrix is the sum of its diagonal elements.

\[
\text{Tr}_M = \sum_{i=1}^{n} m_{ii} = \sum_{i,j=1}^{n} m_{ij} \delta_{ij}.
\]

\textbf{Diagonal Matrix:} A matrix with all elements equal to zero except perhaps for those on the diagonal; elements \( a_{ij} \) with \( a_{ij} = 0 \) if \( i \neq j \).

\textbf{The identity matrix:} A square diagonal matrix \( 1 \) with all 1’s on the diagonal and zeroes elsewhere. Note that \( 1_{n \times n} \) times any \( n \times m \) matrix is that same matrix. \( 1 M = M \), \( M 1 = M \). That is \( 1 \) is the multiplicative identity for matrix multiplication. The matrix \( 1 \) has elements \( \delta_{ij} \) (The Kronecker delta: \( \delta_{ij} = 1 \) if \( i = j \); = 0 otherwise). Alternative notations for \( 1 \) are \( \mathbb{1} \) and \( \mathbb{I} \).

\textbf{Triangular Form:} A matrix with all elements equal to zero either above or below the main diagonal, \( a_{ij} = 0 \) if \( i < j \) or \( a_{ij} = 0 \) if \( i > j \). The diagonal elements form a stair-step leading to the alternative descriptor: \textit{echelon form}. \textit{Upper echelon form} has zeros below the diagonal while \textit{lower
**Echelon form** has zeros above the diagonal.

**Transpose of a Matrix**: The transpose of an $m \times n$ matrix $\mathbf{A}$ with elements $a_{ij}$ is the $n \times m$ matrix $\mathbf{A}^t$ with elements $a'_{ij} = a_{ji}$. That is: the transpose is the matrix in which the rows and columns are interchanged.

**Hermitian Conjugate**: The Hermitian conjugate $\mathbf{A}^\dagger$ (pronounced “A dagger”) of the matrix $\mathbf{A}$ is the complex conjugate of the transpose of $\mathbf{A}$, i.e., $a_{ij}^\dagger = (a_{ji})^\ast$.

Note for the vector space crowd: The matrix formed with inner product elements $\langle B_i | B_j \rangle$, where the set $\{B_1, \ldots, B_i, \ldots, B_n\}$ is a basis set for a vector space, is a Hermitian matrix.

**Symmetric Matrix**: One for which $\mathbf{A} = \mathbf{A}^t$ or, equivalently, $a_{ij} = a_{ji}$.

**Anti-symmetric Matrix**: One for which $\mathbf{A} = -\mathbf{A}^t$ or $a_{ij} = -a_{ji}$. All the diagonal elements of an anti-symmetric matrix are zero. (also called: skew-symmetric)

**Hermitian Matrix**: A matrix $\mathbf{A}$ equal to its Hermitian conjugate $\mathbf{A}^\dagger$, i.e., $a_{ij} = a_{ji}^\ast$.

**Real Hermitian Matrix**: Symmetric

**Singular Matrix**: A square matrix for which no multiplicative inverse exists is a singular matrix. As will be demonstrated, an inverse can be constructed for all square matrices with non-zero determinants. A matrix is singular if its determinant is zero.

---

**Back to the problem of systems of linear algebraic equations**,
\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
\] or \( A x = c \) \[MD.4\]

where both \( x \) and \( c \) are column vectors, \( n \times 1 \) matrices. Given that there is a matrix multiplication, the problem would be solved if a multiplicative inverse for the matrix \( A \) could be found.

**The task:** Find \( A^{-1} \) such that \( A^{-1} A = I \).

\[
A^{-1} A x = I x \Rightarrow x = A^{-1} c \]

[MD.5]

The determinant is to be introduced to establish conditions necessary for \( A^{-1} \) to exist and to provide a method to compute \( A^{-1} \).

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**The Determinant of a Matrix**

The determinant operation on a square matrix yields a scalar value. Its definition and properties are detailed in the following pages. The definition of determinant presented here is based on the permutation symbol. This definition is absolutely equivalent to definitions found in other references, an equivalence that is to be demonstrated. The definition in terms of the permutation symbol is a particularly powerful one, and the permutation symbol is of interest for applications other than the computation of determinants. A short digression follows to introduce the permutation symbol.

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**The Permutation Symbol**

Begin with a list of \( n \) objects \( \{A,B,C,D,\ldots\} \) and prepare a rearrangement of the items in the list; that is a reordered listing in which each symbol in the original list occurs exactly once. Such a list \( \{A,C,B,D,\ldots\} \) is a permutation of the original list \( \{A,B,C,D,\ldots\} \). Permutations are the result of a set of simple nearest-neighbor interchanges performed sequentially on the list of symbols. For example, one interchange of B and C suffices to get from \( \{A,B,C,D,\ldots\} \) to \( \{A,C,B,D,\ldots\} \). This reordering can be reached by a number of interchange patterns, but they all involve an odd number of simple (nearest-neighbor) interchanges. To get from \( \{A,B,C,D,\ldots\} \) to \( \{C,A,B,D,\ldots\} \) one interchange of B and C followed by an interchange of A and C (which are neighbors as a result of the first
interchange) is a possible sequence; there are many others, but all consist of an even number of
interchanges. In fact each reordering or permutation is uniquely even or odd as determined by the
number of nearest-neighbor interchanges required to achieve it. This feature allows a numerical
value \((-1)^P\) to be assigned to each permutation, where \(P\) is the number of simple nearest-neighbor
interchanges required to reach the permutation starting from the reference ordering. That value is -1
for an odd number of interchanges and +1 for an even number of interchanges.

A common notation for the value of a permutation is the permutation symbol \(\epsilon^{ABCD\ldots}_{ijkl\ldots}\) where
\(\{ABCD\ldots\}\) represents the reference symbol list and \(\{ijkl\ldots\}\) is the rearranged list. The symbol
\(\epsilon^{ABCD\ldots}_{ijkl\ldots} = (-1)^P\) where \(P\) is the number of interchanges required to reach the ordering \(\{ijkl\ldots\}\). If
\(\{ijkl\ldots\}\) is not a permutation of the reference list, then \(\epsilon^{ABCD\ldots}_{ijkl\ldots} = 0\). For example, \{2124\} is not a
permutation of \{1234\}. A permutation is a reordering in which each symbol appears exactly once.
For this reason, only reference lists with \(n\) distinct symbols are valid. If a reference list is not
provided, the integers 1,2,3, …, \(n\) are to be the assumed default reference list.

The permutation symbol on three labels \(\epsilon_{ijk} = \epsilon^{123}_{ijk}\) is an incredibly useful notation. It is defined as:

(Alternative nomenclature: The Levi-Civita symbol)

\[
\begin{align*}
\epsilon_{ijk} &= \left\{ \begin{array}{l}
+1 \text{ if } ijk = 123,231,312 \\
-1 \text{ if } ijk = 213,132,321 \\
0 \text{ if two indices equal}
\end{array} \right. \\
\text{Counterclockwise } &\Rightarrow \text{ even, +} \\
\text{Clockwise } &\Rightarrow \text{ odd, -}
\end{align*}
\]

That is:

\[\epsilon_{ijk} = 1 \text{ if } ijk \text{ is an even permutation of } 123,\]
\[\epsilon_{ijk} = -1 \text{ if } ijk \text{ is an odd permutation of } 123, \text{ and} \]
\[\epsilon_{ijk} = 0 \text{ if } ijk \text{ is not a permutation of } 123.\]
A single permutation of 123 involves interchanging any two of the numbers. So the list 213 involves one interchange of 1 and 2 (an odd number) while 231 results by following with an interchange of 1 and 3 for two interchanges and hence an even permutation. Note: this definition can be extended to permutations of four or even \( n \) labels. [For the \textit{special case of only 3 symbols}, the numbers run in CCW order around a circle for the even permutations.]

\textbf{Exercise:} Identify \( xyz \) with 123; show that the cross product \( \vec{C} = \vec{A} \times \vec{B} \) can be represented by the expression below for the \( i^{th} \) component of \( \vec{C} \).

\[
C_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k
\]

In addition to the representation of the cross product above, note that the inner product of \textit{normal} three-dimensional vectors can be represented as:

\[
\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i = \sum_{i,j=1}^{3} A_i B_j \delta_{ij}
\]

\textbf{Summation Convention:} Repeated indices are summed over so often that it is often understood, and the symbol \( \sum_{j,k=1}^{3} \) is omitted. With this understanding, the equation for the cross product would be written as \( C_i = \varepsilon_{ijk} A_j B_k \). This notation is called (Einstein’s) summation convention, and Einstein heralded it as his \textit{greatest accomplishment}.

\textbf{Explicit Summation Notation:} As greatness is a goal not yet attained, explicit summation notation is to be the norm throughout these handouts. Use \textbf{Explicit Summation Notation}.

\textbf{The Summed Product Identity:} A powerful identity follows from the definition of \( \varepsilon_{ijk} \).

\[
\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{i st} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}
\]  \hspace{1cm} [MD.7]

For example, you can use this identity and the representations for cross and inner products above to prove that: \( (\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = A^2 B^2 - (\vec{A} \cdot \vec{B})^2 = A^2 B^2 \left[ 1 - \cos^2 \theta \right] = A^2 B^2 \left[ \sin^2 \theta \right] \).
Extensions to the Permutation Symbol’s Definition

As defined, the permutation symbol yields a value of 1 or –1 if its indices represent a permutation, a re-ordering, of the integers from 1 to \( n \). It is useful to generalize this definition to a reordering a specified set of objects with a specified reference ordering. Consider \( \varepsilon_{ijkl}^{ABCD} \). With 4 symbols, there are 4! (=24) re-orderings. Some sample values are:

\[
\begin{align*}
\varepsilon_{ABCD} &= \varepsilon_{CABD} = \varepsilon_{ACBD} = \varepsilon_{ACDB} = \ldots = 1 \\
\varepsilon_{BACD} &= \varepsilon_{ACBD} = \varepsilon_{DABC} = \varepsilon_{CABD} = \ldots = -1 \\
\varepsilon_{ABAD} &= \varepsilon_{CADB} = \varepsilon_{DDBC} = \varepsilon_{ACBB} = \ldots = 0
\end{align*}
\]

as each symbol must appear exactly once in a permutation or reordering.

In particular, the symbol \( \varepsilon_{134}^{134} \Rightarrow \varepsilon_{341}^{134} = \varepsilon_{413}^{134} = 1 \); \( \varepsilon_{134}^{134} \Rightarrow \varepsilon_{314}^{134} = \varepsilon_{431}^{134} = -1 \), and permutation symbol is zero otherwise.

\[
\text{That is: } \varepsilon_{ijkl}^{ABCD} = (-1)^P \text{ where } P \text{ is the number of interchanges required to return the labels } i, j, k, \ell \text{ to their reference order } A B C D. \text{ If } i, j, k, \ell \text{ is not a reordering of } A B C D \text{ with each symbol appearing once, then } \varepsilon_{ijkl}^{ABCD} = 0.
\]

**How many** permutations are there? With \( n \) symbols each to be listed once, there are \( n \) ways to choose the first, \((n - 1)\) ways to choose the second, \( \ldots \). Multiplying, there are \( n! \) permutations of \( n \) symbols with all symbols included in each listing exactly once.

**Exercise:** Consider the permutation symbol \( \varepsilon_{ijk}^{*#} \).

Give the values of \( \varepsilon_{*#}^{*#} \), \( \varepsilon_{*#}^{*#} \), \( \varepsilon_{*#}^{*#} \), \( \varepsilon_{*#}^{*#} \) and \( \varepsilon_{*#}^{*#} \).

**Reduction in order for permutation symbols:**

For many calculations, the symbol \( \varepsilon_{i,j,k,\ldots,i_n} \) with the first index set to \( N \) needs to be evaluated in terms of the permutation operator for \((n-1)\) labels. In particular,
\[ \epsilon_{N,j,k,...,\iota_{n}} = \epsilon_{N,j,k,...,\iota_{n}}^{123,...,n} = (-1)^{(N-1)} \epsilon_{N,j,k,...,\iota_{N+1}}^{12j,...,(N+1)...n} = (-1)^{(N-1)} \epsilon_{j,k,...,\iota_{N+1}}^{12j,...,(N+1)...n}. \]

\((N-1)\) interchanges are required to bring \(N\) to the first position in the reference list which is equivalent to a factor of \((-1)^{(N-1)}\). Once \(N\) is locked in the first position, the rest is equivalent to the permutations of the remaining \((n-1)\) labels. (An \(n\)-index symbol with the position of one symbol held fixed is equivalent to ±1 times an \((n-1)\)-index permutation symbol.) To bring the result into the ‘standard’ form, it is noted that: \((-1)^{(N-1)} \equiv (-1)^{(1+N)}\).

\[ \epsilon_{N,j,k,...,\iota_{n}} = \epsilon_{N,j,k,...,\iota_{n}}^{123,...,n} = (-1)^{(1+N)} \epsilon_{j,k,...,\iota_{N+1}}^{12j,...,(N+1)...n} \]  

\[ \text{MD.8} \]

### Compound Permutations:

Suppose that one set of \(P_1\) interchanges is applied to the set \(\{ABCD\ldots\}\) to reach the ordering \(\{ijkl\ldots\}\), and then an additional sequence of \(P_2\) interchanges is applied to the list \(\{ijkl\ldots\}\) to reach the final form \(\{stuv\ldots\}\). The number of inter changes to get from \(\{ABCD\ldots\}\) to \(\{stuv\ldots\}\) is \(P_1 + P_2\). The net value of the permutation is \((-1)^{P_1 + P_2}\). Consider the each step. \(\epsilon_{ijkl\ldots}^{ABCD\ldots} = (-1)^{P_1}\) and \(\epsilon_{stuv\ldots}^{ijkl\ldots} = (-1)^{P_2}\) for the overall result \(\epsilon_{stuv\ldots}^{ABCD\ldots} = (-1)^{P_1 + P_2} = \epsilon_{ijkl\ldots}^{ABCD\ldots} \epsilon_{stuv\ldots}^{ijkl\ldots} = (-1)^{P_1}(-1)^{P_2}\). That is: the permutation symbols obey a chained product rule.

\[ \epsilon_{stuv\ldots}^{ABCD\ldots} = \epsilon_{ijkl\ldots}^{ABCD\ldots} \epsilon_{stuv\ldots}^{ijkl\ldots} \]  

\[ \text{MD.9} \]

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**Leibniz, Gottfried Wilhelm**

[b. Leipzig (Germany), July 1, 1646, d. Hanover (Germany), November 14, 1716]

Leibniz is best known for having invented the calculus independently from Newton; much of the notation and vocabulary used today comes from Leibniz, who had a flair for both symbolism and language. He also took the first steps in symbolic logic. The calculating machine Leibniz invented was the first to multiply as well as add and subtract. In physics he contributed to developing the idea of kinetic energy. Here, Leibniz is noted for his contributions to linear algebra including the definition of determinant adopted below.  

[www.answers.com; library of congress](https://www.answers.com)

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**Permutations and Determinants:** The determinant of a matrix is defined for square matrices.

Consider a 3 \(\times\) 3 matrix \(A\) with elements \(a_{ij}\). Following Gottfried Leibniz, the determinant of \(A\) is defined as:

\[ |A| = \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k}. \]  

\[ \text{MD.10} \]
The row indices are written in numerical order, and a factor $\varepsilon_{ijk}$ is included to account for the permutation of the column numbers. *Every term is a product containing exactly one element from each row and one element from each column.* Note that this definition can be generalized to an $n \times n$ matrix in which case, the summations would run from 1 to $n$. **SLOW DOWN!**

**Definition:** Determinant of a $1 \times 1$ matrix.

$$\mathbb{A} = [a_{11}] \quad \text{and} \quad \det \mathbb{A} = |\mathbb{A}| = \sum_{m=1}^{1} \varepsilon_{m} a_{1m} = (1) a_{11} = a_{11}$$

There is only one permutation (order in which to write) a list of the one number 1. For a $1 \times 1$, the determinant has $1! = 1$ terms. *The term contains one element from each row and one element from each column.* Not a very exciting example. Onward to $n = 2$.

**Definition:** Determinant of a $2 \times 2$ matrix.

$$\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \det \mathbb{A} = |\mathbb{A}| = \sum_{m,n=1}^{2} \varepsilon_{mn} a_{1m} a_{2n} = (1) a_{11} a_{22} + (-1) a_{12} a_{21}$$

Here the list $\{1,2\}$ is the reference order so $\varepsilon_{12} = 1$. The list $\{2,1\}$ is one nearest neighbor exchange from the reference order so $\varepsilon_{21} = -1$. For a $2 \times 2$, the determinant has $2! = 2$ terms. *Each term is a product containing one element from each row and one element from each column.*

The $2 \times 2$ case has a geometric representation.

```
= a_{11} a_{22} - a_{12} a_{21}
```

**PLUS** down to the right and **MINUS** down to the left.

**Definition:** Determinant of a $3 \times 3$ matrix.

$$\mathbb{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad \det \mathbb{A} = |\mathbb{A}| = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = (1) a_{11} a_{22} a_{33} + ...$$

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\[
\varepsilon_{ijk} a_{ij} a_{2j} a_{3k} = (l) a_{11} a_{22} a_{33} + (-l) a_{12} a_{21} a_{33} + (l) a_{12} a_{23} a_{31} + \\
(-l) a_{13} a_{22} a_{31} + (l) a_{13} a_{21} a_{32} + (-l) a_{11} a_{23} a_{32}
\]

Here the list \{1,2,3\} is the reference order so \(\varepsilon_{123} = 1\). The list \{2,1,3\} is one nearest neighbor exchange (permutation) from the reference order so \(\varepsilon_{213} = -1\), and so on. In the case of a \(3 \times 3\) matrix, the determinant has \(3! = 6\) terms. Each term in the expansion of the determinant is a product containing one element from each row and one element from each column.

The \(3 \times 3\) case has a geometric representation.

\[
\begin{align*}
\varepsilon_{13} a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{21} a_{32} a_{13} \\
- a_{13} a_{22} a_{31} - a_{12} a_{21} a_{33} - a_{23} a_{32} a_{11}
\end{align*}
\]

PLUS for dominantly down to the right and MINUS for dominantly down to the left.

**Definition:** Determinant of a \(4 \times 4\) matrix or even an \(n \times n\) matrix.

This permutation expansion is getting cumbersome so a new approach is to be adopted. A rule is presented that allows the determinant of an \(n \times n\) matrix to be expressed in terms of the determinants of \((n-1) \times (n-1)\) matrices. The smaller ones have been defined, and the larger ones can be computed directly from the permutation definition or broken down into sums involving the determinants of smaller matrices. This procedure gets a little confusing. Read the general description; advance to the details for a \(4 \times 4\) expansion; finally, read the general description a second time.
In general: \[ | \mathbf{A} | = \sum_{i,j,k,...,n=1}^n (-1)^i a_{ij} a_{2j} ... a_{nj} = \sum_{i,j,k,...,n=1}^n \epsilon_{i,j,k,...,n} a_{ij} a_{2j} ... a_{nj} \] where the sum is over all permutations of the second indices as compared to their numeric reference order \{123 .... \}. For \( n \) labels, there are \( n! \) possible orderings so there are \( n! \) terms in the expansion of the determinant of an \( n \times n \) matrix. The value of the permutation is to be represented by the permutation symbol.

**Def.:** \[ | \mathbf{A} | = \sum_{i,j,k,...,n=1}^n \epsilon_{i,j,k,...,n} a_{ij} a_{2j} ... a_{nj} \] [MD.11]

This mess can be rewritten as a sum across the elements of the first row as

\[
| \mathbf{A} | = a_{11} (-1) ^{1+i} \sum_{j,k,...,n=1}^n \epsilon_{j,k,...,n} a_{2j} ... a_{nj} + a_{12} (-1) ^{1+2} \sum_{j,k,...,n=1}^n \epsilon_{j,k,...,n} a_{2j} ... a_{nj} + ... + a_{1n} (-1) ^{1+n} \sum_{j,k,...,n=1}^n \epsilon_{j,k,...,n} a_{2j} ... a_{nj}
\]

\[
| \mathbf{A} | = \sum_{i=1}^n a_{1i} (-1) ^{1+i} \sum_{j,k,...,n=1}^n \epsilon_{j,k,...,n} a_{2j} ... a_{nj} = \sum_{i=1}^n a_{1i} (-1) ^{1+i} | \mathbf{M}_{1i} | \]

The crucial observations are that the factor \((-1) ^{1+i}\) keeps track of the permutations necessary to account for relocating the \( i^{th} \) index in the permutation symbol in the first position and that the second sum in the expression is the definition of the determinant of the \((n-1) \times (n-1)\) matrix of elements that remain after removing the \(1^{st} \) row and \( i^{th} \) column from the original matrix. The determinant \( \mathbf{M}_{1i} \) is called the **minor** and the combination \((-1) ^{1+i} | \mathbf{M}_{1i} |\) is a scalar value called the **cofactor** \( C_{1i} \) of \( a_{1i} \).

Note that the minor \( \mathbf{M}_{ij} \) is formed by removing the \( i^{th} \) row and \( j^{th} \) column. As an example, he cofactor for \( a_{12} \) involves determinant of the minor \( \mathbf{M}_{12} \), the original matrix after removing the first row and second column.

\[
\begin{pmatrix}
 a_{21} & a_{22} \\
 a_{31} & a_{33}
\end{pmatrix}
\]

\[ \Rightarrow \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} = \mathbf{M}_{12} \]
**This result is huge!** Every reference proves that its definition of a determinant is equivalent to expansion by minors as is the permutation symbol definition used here. All the definitions agree. They are all equally worthy.

**REVIEW:** Permutation symbol manipulations are awe inspiring – more so than is warranted. Work through the details and they become mundane and obvious.

For the case above, the symbol $\varepsilon_{i_1 j_1, i_2 j_2, ..., i_n j_n}$ with the first index set to $N$ needs to be evaluated in terms of the permutation operator for $(n-1)$ labels. In particular,

$$\varepsilon_{N, j_1, ..., j_n} = \varepsilon_{123...,n} = (-1)^{(N-1)} \varepsilon_{N, j_k, ..., j_{(n-1)}} = (-1)^{(N-1)} \varepsilon_{123, (N-1)(N+1), ..., -n}.$$

$(N-1)$ interchanges are required to bring $N$ to the first position in the reference list which is equivalent to a factor of $(-1)^{(N-1)}$. Once $N$ is locked in the first position, the rest is equivalent to the permutations of the remaining $(n-1)$ labels. To bring the result into the ‘normal’ form, it is noted that $(-1)^{(N-1)} = (-1)^{(1+N)}$.

$$\varepsilon_{N, j_k, ..., j_n} = \varepsilon_{123...,n} = (-1)^{(1+N)} \varepsilon_{j_k, ..., j_{(n-1)}} \quad [MD.12]$$

**Expansion of a 4 x 4 by minors with a few details:**

$$|A| = \sum_{i,j,k,l=1}^4 \varepsilon_{i,j,k,l} a_{i1} a_{2j} a_{3k} a_{4\ell} = a_{11} \sum_{j,k,l=1}^4 \varepsilon_{1,j,k,l} a_{2j} a_{3k} a_{4\ell} +$$

$$+ a_{12} \sum_{j,k,l=1}^4 \varepsilon_{12,j,k,l} a_{2j} a_{3k} a_{4\ell} + a_{13} \sum_{j,k,l=1}^4 \varepsilon_{13,j,k,l} a_{2j} a_{3k} a_{4\ell} + a_{14} \sum_{j,k,l=1}^4 \varepsilon_{14,j,k,l} a_{2j} a_{3k} a_{4\ell}$$

The symbol $\varepsilon_{1,j,k,l}^{1234}$ tracks the permutations of the last three symbols given that the first symbol is fixed as 1. The 1 is in its proper place according to the reference ordering so

$$\varepsilon_{1,j,k,l}^{1234} = (+1) \varepsilon_{2,j,k,l}^{234} = (-1)^{1+1} \varepsilon_{3,j,k,l}^{234}.$$ That is one times the permutation value for the rearrangement of the last three indices. Moving on to $\varepsilon_{2,j,k,l}^{1234} = -\varepsilon_{2,j,k,l}^{2134}$. The sign change accounts for the interchange of 1 and 2. Now that the 2 has been moved to the first position, only the permutations of the last the symbols need be tracked. Hence, $\varepsilon_{2,j,k,l}^{1234} = (-1) \varepsilon_{2,j,k,l}^{2134} = (-1) \varepsilon_{2,j,k,l}^{1234} = (-1)^{1+2} \varepsilon_{2,j,k,l}^{1234}$. Continuing, the determinant becomes.
\begin{align*}
&= a_{11} \sum_{j,k,\ell=1}^{4} (-1)^{1+1} \varepsilon_{j k \ell}^{234} a_{2 j} a_{3 k} a_{4 \ell} + a_{12} \sum_{j,k,\ell=1}^{4} (-1)^{1+2} \varepsilon_{j k \ell}^{134} a_{2 j} a_{3 k} a_{4 \ell} + \\
&\quad \quad a_{13} \sum_{j,k,\ell=1}^{4} (-1)^{1+3} \varepsilon_{j k \ell}^{124} a_{2 j} a_{3 k} a_{4 \ell} + a_{14} \sum_{j,k,\ell=1}^{4} (-1)^{1+4} \varepsilon_{j k \ell}^{123} a_{2 j} a_{3 k} a_{4 \ell} \\
&= a_{11} (-1)^{1+1} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{234} a_{2 j} a_{3 k} a_{4 \ell} \right\} + a_{12} (-1)^{1+2} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{134} a_{2 j} a_{3 k} a_{4 \ell} \right\} + \\
&\quad \quad a_{13} (-1)^{1+3} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{124} a_{2 j} a_{3 k} a_{4 \ell} \right\} + a_{14} (-1)^{1+4} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{123} a_{2 j} a_{3 k} a_{4 \ell} \right\}
\end{align*}

Each factor in braces matches the definition for a determinant of a specific matrix. There are extra terms in each sum, but, in each of those cases, the subscript indices are not a permutation of the reference set so the term added is zero. A purist might make this explicit.

\begin{align*}
&= a_{11} (-1)^{1+1} \left\{ \sum_{j,k,\ell=2}^{4} \varepsilon_{j k \ell}^{234} a_{2 j} a_{3 k} a_{4 \ell} \right\} + a_{12} (-1)^{1+2} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{134} a_{2 j} a_{3 k} a_{4 \ell} \right\} + \\
&\quad \quad a_{13} (-1)^{1+3} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{124} a_{2 j} a_{3 k} a_{4 \ell} \right\} + a_{14} (-1)^{1+4} \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{123} a_{2 j} a_{3 k} a_{4 \ell} \right\}
\end{align*}

How were the not-a-permutation-of-the-reference-set terms excluded in each sum above?

For the term multiplied by \(a_{1j}\), sum in braces is the determinant of the matrix is the one that remains after removing row 1 and column \(j\) from the original matrix. That matrix \(\mathbb{M}_{1j}\) is the minor of \(\mathbb{A}\) for the element \(a_{1j}\). Examine the form of the third such factor.

\begin{align*}
\left| \mathbb{M}_{13} \right| &= \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{124} a_{2 j} a_{3 k} a_{4 \ell} \right\} = \left\{ \sum_{j,k,\ell=1}^{4} \varepsilon_{j k \ell}^{124} a_{2 j} a_{3 k} a_{4 \ell} \right\}
\end{align*}

Quick Question: Identify the property of the permutation symbol that 'zeroes out' the terms in the last sum with any of \(\{j, k, \ell\}\) equal to 3.

The elements of \(\mathbb{M}_{13}\) are those of \(\mathbb{A}\) that remain after the first row and third column have been removed. The row labels appear in numerical order and a permutation symbol appropriate for re-
orderings with respect to the reference ordering for the columns is included. By definition, it is the
determinant of the minor, \( |\mathbb{M}_{ij}| \). After some assembly:

\[
|A| = \sum_{j=1}^{4} a_{1j} (-1)^{1+j} |\mathbb{M}_{1j}| = \sum_{j=1}^{4} a_{1j} C_{1j}
\]

This result is the expansion of \( |A| \) in minors of the elements in the first row of \( A \), and the same
result expressed in terms of the cofactors of the elements in the first row. Note that the cofactor
absorbs the factor of \((-1)^{1+j}\) that must be \textit{explicitly} included with the minor.

**Sample calculation of a 3 \times 3 using expansion by minors:**

Expanding along the first row:

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
= a_{11} (-1)^{1+1} \begin{vmatrix}
  a_{22} & a_{23} \\
  a_{32} & a_{33}
\end{vmatrix}
+ a_{12} (-1)^{1+2} \begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix}
+ a_{13} (-1)^{1+3} \begin{vmatrix}
  a_{21} & a_{22} \\
  a_{31} & a_{32}
\end{vmatrix}
\]

\[
= a_{11} (-1) (a_{22} a_{33} - a_{23} a_{32}) + a_{12} (a_{21} a_{33} - a_{23} a_{31}) + a_{13} (-1) (a_{21} a_{32} - a_{22} a_{31})
\]

\[
= a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} + a_{12} a_{21} a_{33} - a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}
\]

This result is correct! Note that the \( ij \) minor is formed by removing the \( i^{th} \) row and \( j^{th} \) column so the
minor for \( a_{12} \) is the original matrix less the first row and second column.

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\Rightarrow
\begin{vmatrix}
  a_{21} & a_{23} \\
  a_{31} & a_{33}
\end{vmatrix}
= \mathbb{M}_{12}
\]

**Exercise:** Use the expansion-by-minors procedure to compute the determinant of the 4 \times 4 identity
matrix. Give the value of the determinant of the \( n \times n \) identity matrix.

**Exercise:** Use the expansion-by-minors procedure to compute the determinant of a 4 \times 4 diagonal
matrix. Express the value if its determinant in terms of the values of elements in the matrix. The
element in the \( i^{th} \) row and \( j^{th} \) column is to be designate \( a_{ij} \).
Properties of the determinant:

Interchanging two columns of a matrix changes the sign of its determinant.

Suppose two columns are interchanged.

\[ |\mathbf{A}| = \sum_{i,j,k,\ldots,n} \epsilon_{i,j,k,\ldots,n} a_{ij} a_{2j} \ldots a_{nj} \]

Interchanging two adjacent columns is equivalent to one permutation or a change of sign. If there are \( n \) columns between the two to be interchanged, then \( n \) interchanges are needed to make them adjacent, one to swap them and \( n \) more to move the swapped column back across the \( n \) separating columns (\( = n + 1 + n \)). It takes an odd permutation to interchange two columns no matter where they are in the matrix. Interchanging two columns of a matrix changes the sign of its determinant.

The determinant of the transpose is equal to determinant of the matrix.

\[ |\mathbf{A}^t| = \sum_{i,j,k,\ldots,n} \epsilon_{i,j,k,\ldots,n} a_{ij}^t a_{2j}^t \ldots a_{nj}^t = \sum_{i,j,k,\ldots,n} \epsilon_{i,j,k,\ldots,n} a_{1i} a_{j2} \ldots a_{jn} \]

where the determinant of the transpose is expressed in terms of the elements of the original matrix.

To get back to the definition of the original matrix, the first index of each \( a_{ij} \) needs to be in numerical order and the term needs to be multiplied by the permutation symbol for the resulting scrambling of the order of the second index. Now as written, \( \epsilon_{i,j,k,\ldots,n} \) is the permutation of the first index and represents the interchanges necessary to bring them into numerical order. If this is done, it also represents the resultant permutation of the second indices that had been in numerical order. That is: it will be exactly the definition of \( |\mathbf{A}| \). Hence \( |\mathbf{A}| = |\mathbf{A}^t| \).

Interchanging two rows of a matrix changes the sign of its determinant. The transpose is equivalent to interchanging the roles of rows and columns. Going to the transpose changes nothing.

To interchange rows: transpose, use the column interchange rule, and transpose back.

Interchanging any two rows in a matrix changes the sign of its determinant.
**Corollary 1:** \( \text{Det} = \text{ZERO for identical rows or columns.} \) If any two rows (or columns) of a matrix are identical, its determinant is zero. If you interchange the two identical rows you get the negative of the original value, hence that value must be zero.

**Corollary 2:** (The Laplace Expansion Theorem) \textbf{An expansion by minors can be made based on the elements of any row or column.} Using the rules for column interchanges and transposes, the expansion in minors becomes:

\[
\begin{vmatrix} A \end{vmatrix} = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \begin{vmatrix} M_{ij} \end{vmatrix} = \sum_{j=1}^{n} a_{ij} \ C_{ij} \quad \text{sum across row } i
\]

\[
\begin{vmatrix} A \end{vmatrix} = \sum_{j=1}^{n} a_{ji} (-1)^{i+j} \begin{vmatrix} M_{ji} \end{vmatrix} = \sum_{j=1}^{n} a_{ji} \ C_{ji} \quad \text{sum down column } i
\]

The first sum is over the elements of row \( i \) and the second is over the elements of column \( i \). In each case \( C_{ij} \) is the cofactor of \( a_{ij} \) which is \((-1)^{i+j} \begin{vmatrix} ij \end{vmatrix}\).

**Rule of Multiples:** If each element in one row (or in one column) of a matrix is multiplied by the same constant \( k \), then the determinant is multiplied by the same constant \( k \). By the definition, every term in the expansion of the determinant contains exactly one element (factor) from each row (or from each column) in the matrix. Each term in the expansion is multiplied by one factor of \( k \). The determinant is the sum of terms and so is also multiplied by \( k \).

\textbf{NOTE: Here the elements of one row are multiplied by } \( k \). \textbf{In contrast } \( k A \) \textbf{ is the matrix in which every element is multiplied by } \( k \) \textbf{ with the result } \( |k A| = k^n |A| \) \textbf{ which follows by applying the rule of multiples } \( n \) \textbf{ times, once for each row.}

**Rule of Sums:** If each element in one row (or in one column) of a matrix is a sum of two terms, then the determinant is the sum of the determinant of the matrix in which that row is filled only with the first terms and the determinant of the matrix in which that row is filled only with the second terms. This result can be established directly from the definition, but the expansion in terms of minors is used as an economy. (illustrated for row \( i \) having sum elements.) Hence the determinant is expanded in minors of that row.
\[ |\mathbf{A}_{a+b}| = \sum_{j=1}^{n} (a_{ij} + b_{ij}) (-1)^{i+j} |\mathbf{M}_{ij}| = \sum_{j=1}^{n} (a_{ij})(-1)^{i+j} |\mathbf{M}_{ij}| + \sum_{j=1}^{n} (b_{ij})(-1)^{i+j} |\mathbf{M}_{ij}| \]

\[ |\mathbf{A}_{a+b}| = |\mathbf{A}_a| + |\mathbf{A}_b|. \]

One row of \( \mathbf{A}_{a+b} \) is the sum of that same row in \( \mathbf{A}_a \) and in \( \mathbf{A}_b \). All other elements are identical.

**Rule of Linear Combinations:** If a row (or a column) of a matrix is a linear combination of the other rows (or columns) of the matrix, the determinant is zero.

Use the Rule of Sums to break the primary determinant into a sum of secondary determinants of matrices each with a row (or column) being a multiple of some other row. Use the rule of multiples to express each secondary determinant as a scalar multiple of the determinant of a matrix in which two rows (or columns) are identical. The determinant of each term vanishes so the sum vanishes.

**Determinant of a Triangular Matrix:** The determinant of a triangular matrix (assume an upper-echelon form; zeroes below the descending diagonal) is the product of its diagonal elements.

Evaluate the determinant using a succession of expansions by minors. Expand by the first column that has one non-zero element \( a_{11} \). The determinant is therefore \( a_{11} |\mathbf{M}_{11}| \). Expand the minor using minors of the second column in which only \( a_{22} \) is non-zero, and so on. If the triangular form has the ascending diagonal as a boundary, the expansion-by-minors procedure must be followed in detail to find the sign of the result. See Expediting the evaluation of determinants in the Tools of the Trade section.

\[ |\mathbf{A}_{\text{upper-triang}}| = a_{11} (-1)^{1+1} \{ a_{22} (-1)^{2+2} \{ a_{33} (-1)^{3+3} \{ ....... \} \} \} = \prod_{i=1}^{n} a_{ii} \]

The product symbol \( \prod_{i=1}^{N} b_i \) represents \( (b_1)(b_2) \ldots (b_N) \), the product if the \( N \) values.

The descending diagonal is implicit as: \( i = j \).

The result follows for a matrix in lower-echelon form by using \( |\mathbf{A}| = |\mathbf{A}^t| \).

**Corollary:** The determinant of a diagonal matrix is the product of its diagonal elements.

**Extension:** The determinant of a matrix block diagonal form is the product of the determinants of the blocks.
**Determinant of a Product of Matrices:** The determinant of the product of two matrices is the product of the determinants of the matrices. \( |A \cdot B| = |A| \cdot |B| \).

Using \((A \cdot B)_{ij} = \left( \sum_{r=1}^{n} a_{ir} b_{rj} \right)\), \( |A \cdot B| = \sum_{i,j,k,m,p=1}^{n} \varepsilon_{ijk}\ldots a_{ir} b_{rj} a_{sm} b_{mj} b_{pk} \ldots \)....

Reordering the finite sums yields \( |A \cdot B| = \sum_{\ell,m,p=1}^{n} a_{\ell r} a_{2m} a_{3p} \ldots \sum_{i,j,k=1}^{n} \varepsilon_{ijk}\ldots b_{ri} b_{mj} b_{pk} \ldots \) which is similar to the definition of \( |A| \cdot |B| \), but without the correct permutation symbols. If the indices \( \ell mp \) are distinct, \( \sum_{i,j,k=1}^{n} \varepsilon_{ijk}\ldots b_{ri} b_{mj} b_{pk} \ldots \) is \( \varepsilon_{\ell mp} \) times \( |B| \). If the indices \( \ell mp \) are not distinct, \( \sum_{i,j,k=1}^{n} \varepsilon_{ijk}\ldots b_{ri} b_{mj} b_{pk} \ldots \) is the determinant of a matrix with two identical rows which is zero by corollary one of the row interchange property. That is: There are contributions only from the terms for which the \( \ell mp \) are distinct and hence represent a permutation. In those cases symbols \( \varepsilon_{\ell mp}\ldots \varepsilon_{\ell mp}\ldots \) can be inserted as it has value \( = (-1)^{P}(-1)^{P}((-1)^{2P}) = 1 \).

\( |A \cdot B| = \sum_{\ell,m,p=1}^{n} \varepsilon_{\ell mp}\ldots a_{\ell r} a_{2m} a_{3p} \ldots \sum_{i,j,k=1}^{n} \varepsilon_{ijk}\ldots b_{ri} b_{mj} b_{pk} \ldots \)

Clearly, \( \sum_{\ell,m,p=1}^{n} \varepsilon_{\ell mp}\ldots a_{\ell r} a_{2m} a_{3p} \ldots = |A| \), and, after a little thought, \( \varepsilon_{\ell mp}\ldots \varepsilon_{ijk}\ldots = (-1)^{P}(-1)^{P} \) is just the value for the permutation that first accounts for the \( P_{1} \) scrambling of the first indices and the \( P_{2} \) scrambling of the second indices of the terms \( (b_{ri} b_{mj} b_{pk} \ldots) \). After \( P_{1} \) simple nearest-neighbor interchanges are applied to place the first indices in numerical order as required to match the definition of a determinant, the second indices are \( P_{1} + P_{2} \) interchanges from the numerical order. Letting \( \varepsilon_{\ell mp}\ldots \varepsilon_{ijk}\ldots = (-1)^{P}(-1)^{P} = \varepsilon_{stu}\ldots \) and realizing that the sum over \( ijk \) actually directs a sum over all permutations as does the sum over \( stu \) with \( \varepsilon_{stu} \), we have:

\( |A \cdot B| = \sum_{\ell,m,p=1}^{n} \varepsilon_{\ell mp}\ldots a_{\ell r} a_{2m} a_{3p} \ldots \sum_{i,j,k=1}^{n} \varepsilon_{ijk}\ldots b_{ri} b_{mj} b_{pk} \ldots \)

\( = \left( \sum_{\ell,m,p=1}^{n} \varepsilon_{\ell mp}\ldots a_{\ell r} a_{2m} a_{3p} \ldots \right) \left( \sum_{s,t,u=1}^{n} \varepsilon_{stu}\ldots b_{rs} b_{tu} b_{pu} \ldots \right) = |A| \cdot |B| \)
**Exercise:** Use the facts that $BB^{-1} = 1$ and that $|\mathbf{I}| = 1$ to express the value of $|B^{-1}|$ in terms of $|B|$. 

**Exercise:** Use a brute force calculation to show that $|A||B| = |AB|$ for the matrices below.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

**Exercise:** Use a brute force calculation to show that $|A||B| = |AB|$ for the matrices below.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

---

**The Multiplicative Inverse of a Matrix $A^{-1}$:**

The determinant provides all the tools necessary to compute the multiplicative inverse of a matrix. Hang on; it's not pretty. The process begins with the right-side inverse.

The determinant of a matrix can be computed as:

$$|A| = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} |M_{ij}| = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{where} \quad C_{ij} = (-1)^{i+j} |M_{ij}|, \text{the cofactor.}$$

The new matrix below is formed using the $ij$ cofactor of $A$ as its $ji$ element. It is the transpose of the matrix of cofactors. Multiply the first row (vector) of $A$ times this matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & \ldots & \ldots & C_{nn} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} C_{1j} & \sum_{j=1}^{n} a_{1j} C_{2j} & \ldots & \sum_{j=1}^{n} a_{1j} C_{nj} \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}.$$
gives $\sum_{j=1}^{n} a_{ij} C_{2j}$ which is the determinant of the full matrix $|A|$ except that $[a_{11} \ a_{12} \ \ldots \ a_{nn}]$ would appear in the first and second rows. The determinant vanishes whenever two rows or columns are identical. That is: the sum $\sum_{j=1}^{n} a_{ij} C_{ij} = |A| \delta_{ij}$. It represents the determinant when the cofactors are matched with the correct row of $A$ and the determinant of a matrix with two identical rows when they are matched with the wrong row. Filling out the rows in $A$.

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & \ldots & \ldots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
C_{11} & C_{21} & \ldots & C_{n1} \\
C_{12} & C_{22} & \ldots & \ldots \\
\vdots & \vdots & \ddots & C_{ij} \\
C_{1n} & \ldots & \ldots & C_{nn}
\end{bmatrix} = |A| \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix} = |A|^{-1}
\]

This matrix formed using the $C_{ij}$'s ad the $ji$ elements has proven useful. It is called the classical adjoint of $A$.

\[
\text{adj } A = [C_{ij}^A]^t
\]

That is: the classical adjoint of $A$ is the transpose of the matrix of cofactors for $A$. An adjoint is a general mathematical term for an entity that works with or is associated with another by a process.

**Inverse of a Matrix:**  \[ \Rightarrow \textit{Celebrate! We have found } A^{-1}. \]

$A^{-1} = \{[C_{ij}^A]^t\}/|A|$, the classical adjoint of $A$ divided by its determinant.

The inverse exists and can be computed by the formula above as long as $|A|$ is non-vanishing. As developed, $A^{-1} = \{[C_{ij}^A]^t\}/|A|$ is the right-side inverse for the matrix $A$. A matrix is said to be singular if its determinant vanishes. If a square matrix is not singular, it has an inverse.

**Exercise:** Find the matrix inverse of an arbitrary $2 \times 2$ matrix. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, find the matrix of cofactors. Find $\text{adj } A = [C_{ij}^A]^t$. Finally, find $A^{-1} = [C_{ij}^A]^t/|A|$. Conclude that:
Show that the proposed inverse is correct by direct calculation.

See the Row Reduction handout for a second method used to compute the inverse matrix.

A right-side inverse is also a left-side inverse.
The classical adjoint and determinant provide $A_R^{-1}$, a right-side inverse for a non-singular square matrix $A$. A non-singular matrix is one for which the determinant is non-zero. The task now is to show that the procedure supplies the inverse, not just the right-side inverse ($A_R^{-1} = A^{-1}$). Matrix multiplication is not commutative so $A A_R^{-1} = 1$ may not ensure that $A_R^{-1} A = 1$.

Given: $A_R^{-1}$ is a right-side inverse for $A$ $\Rightarrow A A_R^{-1} = 1$ and that a right-side inverse exists for $A_R^{-1} = (A_R^{-1})_R^{-1}$. All the matrices are $n \times n$, and all have non-vanishing determinants. A right-side inverse exists for all such matrices and can be computed as the classical adjoint of the original matrix divided by its determinant. By the product property, the determinant of the inverse of a matrix is just the inverse of its determinant. Also assumed: $C 1 = 1$, $C = C$ for all $C$. (See problem 6.)

Given that $A_R^{-1}$ is the right-side inverse of $A$ and that a right-side inverse exists for $A_R^{-1}$, show that $A_R^{-1}$ is a left-side inverse for $A$.

$A A_R^{-1} = 1$ and $A_R^{-1} (A_R^{-1})_R^{-1} = 1$

Pre-multiply by $A_R^{-1}$ and use the associative property of multiplication.

$A_R^{-1} A A_R^{-1} = A R^{-1} (A R^{-1}) = A R^{-1} 1 = A R^{-1}$

Post multiply by $(A_R^{-1})_R^{-1}$.

$A R^{-1} A R^{-1} (A_R^{-1})_R^{-1} = A R^{-1} (A_R^{-1})_R^{-1}$

Hence

$A R^{-1} A = 1$

demonstrating that $A R^{-1}$ is also the left-side inverse for $A$. The $n \times n$ specification ensures that both $A A_R^{-1}$ and $A_R^{-1} A$ have the same dimensions and that the determinant is defined.
**The inverse of a matrix is unique.** Suppose that the matrix $|A| \neq 0$ and that $A$ has two inverses $B$ and $C$ or more generally suppose that the following equation holds.

$$A B = A C$$

then

$$A^{-1} A B = A^{-1} A C \text{ or } B = C$$

In the special case that $AB = AC = 1$, the right-side inverse of $A$ is shown to be unique.

**Sample Calculation:** Finding the inverse of a $3 \times 3$ matrix

Only the results after each step are displayed.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad |A| = -2 \quad (C)_{ij} = C_{ij} = (-1)^{i+j} |M_{ij}|$$

$$C = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{adj} A = C^t = \begin{bmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

$$A^{-1} = (\text{adj} A) / |A| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Exercise: Compute

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$
to verify that the matrix above is $A^{-1}$.

---

**Cramer's Rule:** The solution of systems of linear algebraic equations

The solution of the set of linear algebraic equations: $A x = c$ is $x = A^{-1} c$. 

---

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n \\
\end{bmatrix} = |\mathbf{A}|^{-1} \begin{bmatrix}
    C_{11} & C_{12} & \cdots & C_{1n} \\
    C_{21} & C_{22} & \cdots & C_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    C_{n1} & C_{n2} & \cdots & C_{nn} \\
\end{bmatrix} \begin{bmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_n \\
\end{bmatrix} = |\mathbf{A}|^{-1} \begin{bmatrix}
    \sum_i c_i C_{i1} \\
    \sum_i c_i C_{i2} \\
    \vdots \\
    \sum_i c_i C_{in} \\
\end{bmatrix}
\]

\[
x_j = |\mathbf{A}|^{-1} \sum_{i=1}^n c_i C_{ij}
\]

Comparing with the expressions for expansion in minors and cofactors,

\[
|\mathbf{A}| = \sum_{j=1}^n a_{ij} (-1)^{i+j} |\mathbf{M}_{ij}| = \sum_{j=1}^n a_{ij} C_{ij} \quad \text{sum across row } i
\]

\[
|\mathbf{A}| = \sum_{j=1}^n a_{ij} (-1)^{i+j} |\mathbf{M}_{ji}| = \sum_{j=1}^n a_{ij} C_{ij} \quad \text{sum down column } i
\]

The right hand side of the equation for \(x_j\) has the form of a sum down a column to compute a determinant, but the \(j^{th}\) column vector of matrix elements is replaced by the column vector of constant values (the \(c^\prime\)’s). Hence, we find Cramer’s Rule:

\[
x_1 = |\mathbf{A}|^{-1} \begin{bmatrix}
    c_1 & a_{12} & \cdots & a_{1n} \\
    c_2 & a_{22} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    c_n & \cdots & \cdots & a_{nn} \\
\end{bmatrix}; \quad x_2 = |\mathbf{A}|^{-1} \begin{bmatrix}
    a_{11} & c_1 & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & c_n & \cdots & a_{nn} \\
\end{bmatrix}; \quad x_3 = \ldots \ldots
\]

**Sample Calculation:** Cramer’s Rule for a system of three equations

\[
\begin{align*}
    x + y &= 4 \\
    x - y &= 2 \\
    x + y + z &= 2
\end{align*}
\]

or

\[
\begin{bmatrix}
    1 & 1 & 0 \\
    1 & -1 & 0 \\
    1 & 1 & 1 \\
\end{bmatrix} \begin{bmatrix}
    x \\
    y \\
    z \\
\end{bmatrix} = \begin{bmatrix}
    4 \\
    2 \\
    2 \\
\end{bmatrix} \Rightarrow (x, y, z) = (3, 1, -2)
\]
Cramer’s Rule Summary:

Linear Equations: We introduced the concept of a matrix and hence the determinant by means of consideration of a system of linear equations. Let us now return to that consideration as an example of the use of determinants.

Consider the set of \( n \) non-homogeneous linear equations in \( n \) unknowns.

\[
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\
\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = c_n
\]

This set of equations can be expressed in matrix form as

\[
\begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \ldots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix} =
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{bmatrix}
\]

[MD.14]
Let $A$ be the $n \times n$ matrix of the coefficients. This system of $n$ equations in $n$ unknowns always has a unique solution if the determinant of the coefficient matrix is not zero ($|A| \neq 0$). If this is the case, Cramer's rule computes each unknown $x_n$ by replacing the $n^{th}$ column in $A$ with the values of $c_n$, computing the determinant of this new matrix, and dividing that result by the value of $|A|$. As an example, the value for $x_2$ will be:

$$x_1 = \left| \begin{array}{c} c_1 & a_{12} & \cdots & a_{1n} \\ c_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_n & a_{n2} & \cdots & a_{nn} \end{array} \right|^{-1} \left| \begin{array}{c} c_1 \\ c_2 \\ \vdots \\ c_n \end{array} \right|; \quad x_2 = \left| \begin{array}{c} a_{11} & c_1 & \cdots & a_{1n} \\ a_{21} & c_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & c_n & \cdots & a_{nn} \end{array} \right|^{-1} \left| \begin{array}{c} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{array} \right|; \quad x_3 = \ldots .$$

Analogous expressions represent the other unknowns, $x_i$.

In the case $|A| = 0$, not all the equations are independent so Cramer's rule fails to provide unique values for the unknowns. The rank of $A$, $r$, is the dimension of the largest sub-matrix of $A$ with non-zero determinant. Expect that one can solve for $r$ of the unknowns in terms of the constants $c_n$ and the other $n - r$ unknowns.

In the event that you have a set of homogeneous equations (that is, all of the $c_n = 0$) then if $|A| \neq 0$ the only possible solution is the trivial solution where all $x_n = 0$. If, however, $|A| = 0$, a non-trivial solution may exist. The last case is of very special interest for eigenvalue problems such as finding the normal frequencies and modes of a system of couple oscillators.

Cramer's rule provides unique solutions when they exist, and it provides a test ($|A| = 0$) for the case that the system of equations is underdetermined.

---

**Rank of a Matrix**: The rank of a matrix is the number of independent equations represented by the rows or columns of the matrix. The two agree. Given an $n \times m$ matrix, rows and/or columns may be excluded to form included square ($k \times k$) matrices. The largest included square matrix with non-zero determinant is designated as $(r \times r)$ where $r$ is the rank of the original matrix. If the rank of a square $(n \times n)$ matrix is not equal to its size $n$, its determinant is zero, and it represents only $r$ independent equations.
Functions of a Matrix Argument:

A function of an \( n \times n \) matrix is defined in terms of the power series expansion for the function.

\[
f[x] = a_0 + a_1 x + a_2 x^2 + \ldots + a_i x^i + \ldots
\]

The matrix replaces \( x \) and the powers are computed using matrix multiplication. This procedure works because products of \( n \times n \) matrices are also \( n \times n \) matrices.

\[
f[M] = a_0 I + a_1 M + a_2 M M + \ldots + a_i (M^i) + \ldots
\]

Obviously, this process becomes tedious, and relief is sought. If the matrix \( M \) is diagonal with diagonal elements \( m_{ii} = \lambda_i \), the powers become trivial!

\[
[M^n]_{ij} = (\lambda_i)^n \delta_{ij}
\]

The \( n^{th} \) power of a diagonal matrix is just a matrix with the \( n^{th} \) powers of the original diagonal elements on its diagonal.

STOP HERE. SKIP TO TOOLS OF THE TRADE

Insanely irrelevant example: (Skip this section unless specifically advised to read it.)

The Linear Transformations handout discusses the general time development operation for the space of a single simple harmonic oscillator satisfying the equation:

\[
\frac{d^2 S}{dt^2} + S = 0. \text{ Note that } \omega = 1 \text{ for this example.}
\]

The value of the function at \( t + \Delta \) is to be extracted from \( S(t) \) the solution at time \( t \) by evaluating its Taylor's series:

\[
f(t + \Delta) = f(t) + \frac{df}{dt} \Delta + \frac{1}{2!} \frac{d^2 f}{dt^2} \Delta^2 + \ldots + \frac{1}{n!} \frac{d^n f}{dt^n} \Delta^n + \ldots
\]

For this space, two possible time dependence basis sets are examined.

Standard basis: \( \{ \hat{e}_1(t) = \sqrt{2} \cos(t) \; ; \; \hat{e}_2(t) = \sqrt{2} \sin(t) \} \)

Alternative basis: \( \{ \hat{b}_1(t) = e^{it} \; ; \; \hat{b}_2(t) = e^{-it} \} \)

The operator \( \frac{d}{dt} \) can be represented for each basis set.
\[
\frac{d}{dt} \rightarrow \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \text{ std basis} \quad \frac{d}{dt} \rightarrow \begin{bmatrix}
i & 0 \\
0 & -i
\end{bmatrix} \text{ alternative}
\]

The time-development operator has a matrix representation $\hat{T}_b$ in the standard basis and a representation $\hat{T}_b$ in the alternative basis. Each acts on a state vector valid at a time $t$ in its basis to generate the form of the state vector at time $t + \Delta t$.

**Notation:** $f(t) = a \sqrt{2} \cos(t) + b \sqrt{2} \sin(t) \rightarrow \begin{bmatrix} a \\ b \end{bmatrix}$ where $\begin{bmatrix} a \\ b \end{bmatrix}$ is the column vector that represents $f(t)$ in the standard basis. $\frac{df}{dt} = -a \sqrt{2} \sin(t) + b \sqrt{2} \cos(t)$ which would be represented by $\begin{bmatrix} b \\ -a \end{bmatrix}$. Hence the action of the operator $\frac{d}{dt}$ is represented by the matrix \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]
in the standard basis. \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ -a \end{bmatrix} \rightarrow -a \sqrt{2} \sin(t) + b \sqrt{2} \cos(t)
\]

**Exercise:** Show that $\frac{d}{dt} \rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ in the alternative basis representation.

Both cases are tractable, but only the alternative basis provides the diagonal form that simplifies the computation. Use the Taylor's series expansion with the time derivatives represented by matrix multiplications $\frac{d}{dt} \rightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. The matrix operator $\hat{T}_b$ is the time-development operator as expressed in the alternative basis. It acts on a state vector valid at a time $t$ in the $b$-basis to generate the form of the state vector at time $t + \Delta t$.

\[
\hat{T}_b \left( \Delta \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Delta^0 + \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \Delta^1 + \frac{1}{2!} \begin{bmatrix} i^2 & 0 \\ 0 & (-i)^2 \end{bmatrix} \Delta^2 + \cdots + \frac{1}{n!} \begin{bmatrix} i^n & 0 \\ 0 & (-i)^n \end{bmatrix} \Delta^n + \cdots
\]

\[
= \begin{bmatrix}
(i\Delta)^0 & 0 \\
0 & (-i\Delta)^0
\end{bmatrix} + \begin{bmatrix}
(i\Delta)^1 & 0 \\
0 & (-i\Delta)^1
\end{bmatrix} + \frac{1}{2!} \begin{bmatrix}
(i\Delta)^2 & 0 \\
0 & (-i\Delta)^2
\end{bmatrix} + \cdots + \frac{1}{n!} \begin{bmatrix}
(i\Delta)^n & 0 \\
0 & (-i\Delta)^n
\end{bmatrix} + \cdots
\]
In the alternative basis, a state of the oscillator is:

\[ |S(t)\rangle = ae^{i\omega t} + be^{-i\omega t} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} \]

The column vector is the matrix representation of the state in the alternative basis representation.

\[
|S(t + \Delta)\rangle = \hat{T}(\Delta) |S(t)\rangle \Rightarrow \hat{T}(\Delta) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} e^{i\Delta} & 0 \\ 0 & e^{-i\Delta} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a e^{i\Delta} \\ b e^{-i\Delta} \end{bmatrix}
\]

\[
\begin{bmatrix} a e^{i\Delta} \\ b e^{-i\Delta} \end{bmatrix} \Rightarrow a e^{i\Delta} e^{i\omega t} + b e^{-i\Delta} e^{-i\omega t} = a e^{i\omega(t+\Delta)} + b e^{-i\omega(t+\Delta)} = |S(t + \Delta)\rangle
\]

The time development operator propagates the state of the oscillator from its state at time \( t \) to that at time \( t + \Delta \).

**Tools of the Trade:**

**Alternative geometric expansion of a 3 x 3 determinant:**

Expanding along the first row, the expansion of minors method yields:

\[
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

A geometric scheme rapidly reproduces this result.

\[
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\]

This pattern can be read off the figure below in which the entries from rows two and three of the first two columns are recopied to the right of the matrix. The resultant interchanges subsume the sign changes and each contribution from the groupings illustrated below contributes with a positive sign.
\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{21} a_{22} a_{23} \Rightarrow
\begin{vmatrix}
  a_{11} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix}
\]

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{21} a_{22} a_{31} \Rightarrow
\begin{vmatrix}
  a_{12} & a_{23} & a_{21} \\
  a_{33} & a_{31}
\end{vmatrix}
\]

\[
\begin{vmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{vmatrix} = a_{21} a_{22} a_{31} \Rightarrow
\begin{vmatrix}
  a_{13} & a_{22} \\
  a_{31} & a_{32}
\end{vmatrix}
\]

The same trick works for the cross product expressed as a determinant.

\[
\vec{A} \times \vec{B} = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  A_x & A_y & A_z \\
  B_x & B_y & B_z
\end{vmatrix} = A_y B_z - A_b B_x + \hat{k} A_x B_y = i A_y A_z + j A_z A_x + k A_x A_y
\]

A second alternative geometric scheme recopies the lower left block of four elements to the right of the array as done above and, in addition, copies the lower right block of four elements to the left of the matrix. Products starting with each element in the first row and running down to the right are given positive signs. Products starting with each element in the first row and running down to the left are given negative signs.

**The evaluation of determinants can be expedited** by exploiting the properties of determinants:

Only the adding a row rule and the expansion by minors rule are to be used here, but others properties such as that for removing a factor from a row or from every element can be exploited as well. Consider

\[
\begin{vmatrix}
  1 & 2 & 3 \\
  2 & 4 & 3 \\
  4 & 5 & 6
\end{vmatrix} = \begin{vmatrix}
  1 & 2 & 3 \\
  1 & 2 & 0 \\
  4 & 5 & 6
\end{vmatrix} = \begin{vmatrix}
  1 & 2 \\
  4 & 5
\end{vmatrix} = -9
\]

The same trick works for the cross product expressed as a determinant.

\[
\vec{A} \times \vec{B} = \begin{vmatrix}
  \hat{i} & \hat{j} & \hat{k} \\
  A_x & A_y & A_z \\
  B_x & B_y & B_z
\end{vmatrix} = A_y B_z - A_b B_x + \hat{k} A_x B_y = i A_y A_z + j A_z A_x + k A_x A_y
\]
First, row 1 was subtracted from row 2 (adding one row to another does not change the determinant). Next, row 2 was subtracted from row 1. The resulting matrix is to be expanded by minors of the sparse first row. Continue by subtracting (the third row multiplied by one-fourth) from the second row.

\[
\begin{vmatrix}
0 & 0 & 3 \\
1 & 2 & 0 \\
4 & 5 & 6
\end{vmatrix} = \begin{vmatrix}
0 & 0 & 3 \\
0 & 3/4 & -3/2 \\
4 & 5 & 6
\end{vmatrix} = (3)(-3/4)(4) = -9
\]

These procedures lead to a matrix with zeroes above the ascending diagonal. The expansion of minor rule is applied twice being careful to include the correct factors of the form: \((-1)^{i+j}\).

\[
\begin{vmatrix}
0 & 0 & 3 \\
0 & 3/4 & -3/2 \\
4 & 5 & 6
\end{vmatrix} = (-1)^{i+j} (3) \begin{vmatrix}
0 & 3/4 \\
4 & 5
\end{vmatrix} = (-1)^{i+j} (3) \left[ (-1)^{i+j} (3/4) \right] (4) = -9
\]

If the matrix is transformed to a form with all zeroes above (or below) the descending diagonal, the determinant is equal to the product of the elements along the diagonal. That is: all the sign factors from the cofactors are of the form

\((-1)^{i+j} = (-1)^{i+j} = (-1)^2 = +1\)

The determinant of a matrix in (descending diagonal) triangular form is the product of the diagonal elements.

**Exercise:** Consider a square \((n \times n)\) matrix in triangular form with respect to its ascending diagonal. The factor \((-1)^{i+j}\) for the upper corner is \((-1)^{1+n}\). Move down the diagonal to row two. Express \((-1)^{i+j}\) for the new \((n-1) \times (n-1)\) minor. Verify the form: \((-1)^{1+(n-1)}\). Continue to the lower-left corner. Propose a final expression for the determinant as a product of the elements along the ascending diagonal for an ascending diagonal triangular matrix.

GUESS: \(\bigg| A_{\text{ascending-triangular}} \bigg| = \prod_{i=1}^{n} a_{(n+1-i),i} \oplus \begin{cases} +1 & \text{if } n = 4p \text{ or } 4p+1 \\ -1 & \text{if } n = 4p+2 \text{ or } 4p+3 \end{cases}\)

**Summed Product Identity for the Permutation Symbol:**

\[
\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks} \quad \text{Note: Summations are EXPLICIT.}
\]
This identity can be established by a brute force calculation of the 81 \((3^4)\) possible values for \(jkst\).

One can motivate the result by realizing that in the sum over \(i\), there must be at least one instance in which \(i\) is not equal to either \(s\) or \(t\). In this instance, \(\varepsilon_{ist}\) can be non-zero. In order that \(\varepsilon_{ijk}\) also be non-zero, \(ijk\) must be a permutation of \(123\), and, since \(i\) is set, either \(j = s\) and \(k = t\) or \(j = t\) and \(k = s\). Also, we must have \(s \neq t\) in order that \(ist\) be a permutation. (No index value can be repeated.) In the case \(j = s\) and \(k = t\), then both \(\varepsilon_{ijk}\) and \(\varepsilon_{ist}\) are equal to either plus one or minus one (if they are non-zero) so that the product is plus one. In the case \(j = t\) and \(k = s\), then \(\varepsilon_{ijk}\) and \(\varepsilon_{ist}\) differ by one interchange and so have opposite sign (if they are non-zero) making their product minus one.

Agrees with:

\[
\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ist} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}
\]

In the cases for which \(j = s\), \(k = t\), \(j = t\) and \(k = s\) at least one index value is repeated so that it does not represent a permutation yielding \(\varepsilon_{ijk} = \varepsilon_{ist} = 0\).

**Exercise:** Show that \(\delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks} = 0\) if \(s = t\) or \(j = k\).

A more formal development follows from considering the special determinant (See Lewis and Ward; also check problem 46.).

\[
\begin{vmatrix}
\delta_{\ell t} & \delta_{\ell m} & \delta_{\ell n} \\
\delta_{jt} & \delta_{jm} & \delta_{jn} \\
\delta_{kt} & \delta_{km} & \delta_{kn}
\end{vmatrix}
= P_{ijk} \varepsilon_{\ell mn} = c \varepsilon_{ijk} \varepsilon_{\ell mn}
\]

The value of the determinant changes sign with each simple nearest-neighbor interchange of columns as does the permutation symbol with an index interchange. Hence the determinant is some function \(P_{ijk}\) of the indices \(i, j, k\) times the permutation symbol on the labels \(\ell, m, n\). As the value of the determinant also changes sign with each simple nearest-neighbor interchange of rows, the determinant is also proportional to \(\varepsilon_{\ell mn}\). The proportionality constant is established by choosing the
values \{i, j, k\} and \{\ell, m, n\} = \{1,2,3\}. With these choices, the determinant and hence \( c = 1 \). Next,

\[
\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{ist} \text{ is computed.}
\]

Expanding by minors of the first row and collecting terms:

\[
\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{inm} = \sum_{i=1}^{3} \begin{vmatrix}
\delta_{ji} & \delta_{jm} & \delta_{jn} \\
\delta_{ki} & \delta_{km} & \delta_{kn} \\
\end{vmatrix} - \sum_{i=1}^{3} \begin{vmatrix}
\delta_{ji} & \delta_{jn} & \delta_{jm} \\
\delta_{ki} & \delta_{kn} & \delta_{km} \\
\end{vmatrix} + \sum_{i=1}^{3} \begin{vmatrix}
\delta_{ji} & \delta_{jm} & \delta_{jn} \\
\delta_{ki} & \delta_{km} & \delta_{kn} \\
\end{vmatrix}
\]

\[\text{[MD.15]}\]

\[
\sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{inn} = 3 \left[ \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \right] - \sum_{i=1}^{3} \begin{vmatrix}
\delta_{ji} & \delta_{kn} & \delta_{jm} \\
\delta_{ki} & \delta_{km} & \delta_{jn} \\
\end{vmatrix} + \sum_{i=1}^{3} \begin{vmatrix}
\delta_{ji} & \delta_{kn} & \delta_{jm} \\
\delta_{ki} & \delta_{km} & \delta_{jn} \\
\end{vmatrix}
\]

\[\Rightarrow \sum_{i=1}^{3} \varepsilon_{ijk} \varepsilon_{inn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \]

\[\text{[MD.16]}\]

The equation below counts the number of permutations that exist for three labels.

\[
\sum_{i,j,k=1}^{3} \varepsilon_{ijk} \varepsilon_{ikj} = \sum_{j,k=1}^{3} \delta_{ji} \delta_{ik} - \sum_{j,k=1}^{3} \delta_{jk} \delta_{kj} = (3)(3) - \sum_{k=1}^{3} \delta_{kk} = 9 - 3 = 6 = 3!
\]

An alternative derivation of the product rule for permutation symbols begins by noting that:

\[
\varepsilon_{ijk} = \begin{vmatrix}
\hat{e}_i \\
\hat{e}_j \\
\hat{e}_k \\
\end{vmatrix} = \begin{vmatrix}
\delta_{i1} & \delta_{i2} & \delta_{i3} \\
\delta_{j1} & \delta_{j2} & \delta_{j3} \\
\delta_{k1} & \delta_{k2} & \delta_{k3} \\
\end{vmatrix}
\]

Left: coordinate directions as rows. Right: Coordinate directions as columns.

That is: \( \hat{e}_1 = \{100\}, \hat{e}_2 = \{010\}, \ldots \). In this mode, \( \varepsilon_{123} \) is the determinant of the identity or 1. The \( \varepsilon_{ijk} \) character follows because the sign changes for each simple interchange of labels, and the result is zero if two or more labels have the same value. The result matches the definition of \( \varepsilon_{ijk} \).

The product of the determinants is the determinant of the product of the matrices.
Expanding by minors, $\varepsilon_{ijk} \varepsilon_{imn} = \delta_{ijm} \delta_{kn} - \delta_{ijn} \delta_{km} + \delta_{jin} \delta_{km}$. This approach now merges with the previous one at [MD.15] when summed over $i$.

### The Pauli Matrices:

Suppose that a set of $2 \times 2$ matrices is sought such that any $2 \times 2$ matrix can be represented as a sum of the members of that set. Clearly the set must have at least four members as $2 \times 2$ matrices have four independent elements. One choice that could be made is:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

This set of matrices is not very exciting. Spice it up; add the requirement that each member of the set be Hermitian (equal to the complex conjugate of its transpose). The simplest matrix that has imaginary elements that meets this requirement is:

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

An independent off-diagonal matrix needs to have elements that are equal rather than the negative of one another. Equal off-diagonal elements and Hermitian restricts the elements to be real.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Following the equal and negative scheme for the on diagonal matrices, the remaining members are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This particular set of $2 \times 2$ matrices is the set of Pauli matrices. They have assigned labels:

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \ \mathbb{x} = \mathbb{1}; \ \mathbb{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ \mathbb{z} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}; \ \mathbb{z}_{+} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \ \mathbb{z}_{-} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The first matrix is the identity, and the final three are the Pauli matrices. The set of the four matrices is a basis for the collection of all $2 \times 2$ matrices.
\[
\begin{bmatrix}
  a & b \\
  c & d \\
\end{bmatrix} = \left(\frac{a+d}{2}\right)\begin{bmatrix} 1 & 0 \\
  0 & 1 \\
\end{bmatrix} + \left(\frac{b+c}{2}\right)\begin{bmatrix} 0 & 1 \\
  1 & 0 \\
\end{bmatrix} + \left(\frac{c-b}{2i}\right)\begin{bmatrix} 0 & -i \\
  i & 0 \\
\end{bmatrix} + \left(\frac{a-d}{2}\right)\begin{bmatrix} 1 & 0 \\
  0 & -1 \\
\end{bmatrix}
\]

An arbitrary 2 x 2 matrix can be represented as a linear combination of the four matrices, but not as linear combination of any set with fewer than four members. (A basis must be complete in the sense that all elements (matrices) of interest can be represented as linear combinations of the members of the set and economical in the sense that every member is necessary. If even one member of the set is removed then there will be at least one 2 x 2 matrix that cannot be represented as a linear combination of the remaining members.)

**Why are the Pauli matrices introduced?**

The Pauli matrices are introduced to act on two-row column matrices. Matrices of the forms:

\[
\begin{bmatrix}
  a \\
  b \\
\end{bmatrix}, \begin{bmatrix} 1 \\
  0 \\
\end{bmatrix} and \begin{bmatrix} 0 \\
  1 \\
\end{bmatrix}.
\]

The column vectors \begin{bmatrix} 1 \\
  0 \\
\end{bmatrix} and \begin{bmatrix} 0 \\
  1 \\
\end{bmatrix} are a basis set for all the \begin{bmatrix} a \\
  b \\
\end{bmatrix}. The column vectors \begin{bmatrix} 1 \\
  0 \\
\end{bmatrix} and \begin{bmatrix} 0 \\
  1 \\
\end{bmatrix} are to be call UP and DOWN.

If you are not familiar with modern physics, skip paragraph.

In quantum mechanics, the Pauli matrices add the flexibility that allows a wavefunction to represent the spin character of an electron. A two row column vector is appended to a function of position and time.

\[
|\Psi\rangle = \psi(\vec{r}, t) \begin{bmatrix} a \\
  b \\
\end{bmatrix}
\]

The column vector \begin{bmatrix} 1 \\
  0 \\
\end{bmatrix} represents spin up while \begin{bmatrix} 0 \\
  1 \\
\end{bmatrix} represents spin down. The spin part is to be normalized independently so

\[
\begin{bmatrix} a \\
  b \\
\end{bmatrix}^\dagger \begin{bmatrix} a \\
  b \\
\end{bmatrix} = \begin{bmatrix} a^* & b^* \\
  a & b \\
\end{bmatrix} = a^* + b^* = 1
\]

**Special properties of the Pauli matrices:**

\[
\sigma_1 \sigma_1 = \sigma_2 \sigma_2 = \sigma_3 \sigma_3 = 1
\]
The determinants are each of the three Pauli matrices is – 1. The determinant of the identity is, of course, equal to one.

\[ |\sigma_1| = |\sigma_2| = |\sigma_3| = 1 \]

**Actions of the Pauli Set:**

\[ \sigma_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \sigma_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

The operator \( \sigma_x \) lowers the UP state to DOWN and raises the DOWN state to UP.

\[ \sigma_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \sigma_y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

The operator \( \sigma_y \) lowers the UP state to \( i \) times the DOWN and raises the DOWN state to \( -i \) times UP.

\[ \sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

The operator \( \sigma_z \) on the UP state to 1 times the UP and, on the DOWN state, returns – 1 times the DOWN state. The operator \( \sigma_z \) measures the up-ness or down-ness of the state.

The combination of operators \( \sigma_+ = \sigma_x + i \sigma_y \) annihilates UP and raises DOWN to 2 times UP.

\[ \sigma_+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \sigma_+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

The operator \( \sigma_+ \) is called the raising operator.

**Exercise:** Find the action of \( \sigma_- = \sigma_x + i \sigma_y \) on the states UP and DOWN. Propose a name for \( \sigma_- \).

**Exponentials of Matrices:**

The results of this section should be reviewed when one studies quantum mechanics.

Show that: \( (e^A)^t = e^{A^t} \) We know that \( (AB)^t = B^t A^t \), so \( (A^n)^t = (A^t)^n \).

\[ (e^A)^t = (1 + A + \ldots + \frac{1}{n!} A^n + \ldots)^t = (1 + A^t + \ldots + \frac{1}{n!} (A^t)^n + \ldots) = e^{A^t} \]

Show that: \( \text{Tr}[A^t] = \text{Tr}[UAU^{-1}] \) if \( U \) is invertible. All matrices \( n \times n \).
\[
\text{Tr}[A'] = \sum_i A'_i = \sum_{i,k,m} U_{ik} A_{km} U^{-1}_{mi} = \sum_{i,k,m} U_{mi}^{-1} U_{ik} A_{km} = \sum_{k,m} \delta_{mk} A_{km} = \sum_m A_{mm} = \text{Tr}[A]
\]

**Show that:** \(\text{Det}(e^A) = e^{\text{Tr}[A]}\). Assume that there is a unitary matrix that diagonalizes \(A\). \(A' = UAU^{-1}\).

If a square matrix is diagonal, then its determinant is the product of its diagonal elements. Further, then the \(n\)th power of the matrix is just a diagonal matrix with the \(n\)th power of the of the original \(A_{kk}\) element in the \(kk\) position and, of course, zeroes off the diagonal. Recall that \(M = \text{det}[U M U^{-1}]\).

\[
\text{Det}(e^A) = \text{Det}(UE^A U^{-1}) = \text{Det}( (1 + \sum A AA \cdots + (U A U^{-1}U A U^{-1} \cdots U A U^{-1})/n! + \ldots))
\]

\[
\text{Det}(e^A) = \text{Det}( (1 + A' + \ldots + (A')^n/n! + \ldots))
\]

\[
\begin{bmatrix}
1 + A_{11} + \ldots \frac{A_1^n}{n!} & 0 & 0 & 0 \\
0 & 1 + A_{22} + \ldots \frac{A_2^n}{n!} & 0 & 0 \\
0 & 0 & 1 + A_{33} + \ldots \frac{A_3^n}{n!} & 0 \\
0 & 0 & 0 & 1 + A_{nn} + \ldots \frac{A_n^n}{n!}
\end{bmatrix}
\]

\[
\text{Det}(e^A) = \begin{bmatrix} e^{A_1} & 0 & 0 & 0 \\
0 & e^{A_2} & 0 & 0 \\
0 & 0 & e^{A_3} & 0 \\
0 & 0 & 0 & e^{A_n}
\end{bmatrix} = e^{\text{Tr}[A']} = e^{\text{Tr}[A]}
\]

When one considers identities for functions of matrices, the matrix counterparts of scalar identities differ because matrix multiplication is not commutative. That is: \(A B \neq B A\). We define the commutator of \(A\) and \(B\) as \([A, B] = AB - BA\).

The Campbell-Baker-Hausdorff formula states that: \(e^A e^B = e^{A + B + C}\) where \(C\) is \(1/2\)\([A, B]\) plus higher order commutators of \(A\) and \(B\). In the case the higher order commutators vanish, \(e^A e^B = e^{A + B + [A, B]}\).

The methods presented provide strategies for issues in calculus.

\[
\left[\frac{d}{dx}, x\right]f(x) = \frac{d}{dx}(xf(x)) - (x \frac{df}{dx}) = f(x) + x \frac{df}{dx} - (x \frac{df}{dx}) = f(x).
\]

We conclude that the operation \([\frac{d}{dx}, x]\) has the same action on an arbitrary function of \(x\) as multiplying by 1.

We conclude that \([\frac{d}{dx}, x] = 1\) and that the operations of taking a derivative with respect to \(x\) and multiplying by \(x\) do not commute. Again, the experience gained here will help us deal with that issue.
**Limited Proof:** Only the case that higher order commutators vanish is to be treated here. Assume \([[A, B], A] \text{ and } [[A, B], B] = 0]]

**Exercise:** Consider the actions of \([[\frac{d}{dx}, x], x] \text{ and } [[\frac{d}{dx}, x], \frac{d}{dx}]\) on an arbitrary function of \(x\). What to you conclude are the values of those double commutators?

---

**Warmup Problems:**

WUP1 a.) \(A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\), \(C = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \end{bmatrix}\) and \(D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}\).

Which of the following products is defined? \(\{ AC, AD, DC \}\).

b.) \(M = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & -5 \end{bmatrix} \); \(M^t = \)

c.) Compute the trace of \(M\).

Given: \(A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 2 \end{bmatrix}\), \(B = \begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}\), \(C = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\), \(D = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}\).

d.) Compute \(AB\).

e.) Compute \(C + D\).

f.) Compute and compare \(CD\) and \(DC\) for \(C = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}\); \(D = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}\). Comment.
WUP2. Identify $xyz$ with $123$; show that the cross product $\vec{C} = \vec{A} \times \vec{B}$ can be represented by the expression below for the $i^{th}$ component of $\vec{C}$. \(\Rightarrow\) Write it out in detail and compare with another definition of the cross product.

\[ C_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} A_j B_k \]

<table>
<thead>
<tr>
<th>Permutation Circle</th>
<th>+</th>
<th>-</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{ijk}$</td>
<td>+1 if $ijk = 123, 231, 312$</td>
<td>-1 if $ijk = 213, 132, 321$</td>
<td>0 if two indices equal</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Counterclockwise $\Rightarrow$ even, +
Clockwise $\Rightarrow$ odd, -

That is: $\varepsilon_{ijk} = 1$ if $ijk$ is an even permutation of 123,
$\varepsilon_{ijk} = -1$ if $ijk$ is an odd permutation of 123, and $\varepsilon_{ijk} = 0$ if $ijk$ is not a permutation of 123.

[MD.17]

WUP3. In addition to the representation of the cross product in A10, note that the inner product of normal three-dimensional vectors can be represented as:

\[ \vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i B_i = \sum_{i,j=1}^{3} A_i B_j \delta_{ij} \]

Explicitly expand the sums $\sum_{i=1}^{3} A_i B_i$ and $\sum_{i,j=1}^{3} A_i B_j \delta_{ij}$. Note that $\delta_{ij}$ is the Kronecker delta. Search the MD handout for Kronecker delta.

WUP4. Expand each sum below to display all its terms and then evaluate the sum.

a.) $\sum_{m=1}^{5} m \delta_{3m}$

b.) $\sum_{m=1}^{5} m \delta_{km}$ where $k$ is 3.

c.) $\sum_{m=1}^{5} m \delta_{km}$ where $k$ is an integer greater than zero and less than six.
d.) \( \sum_{m=1}^{5} m \delta_{km} \) where \( k \) is an integer greater than six.

e.) \( \sum_{m=1}^{5} f(m) \delta_{km} \) where \( k \) is an integer greater than zero and less than six and \( f(m) \) is an arbitrary function of the integer argument \( m \). Evaluate for \( f(m) = m^2 \sin(\frac{1}{2} m \pi) \).

WUP5. Product Properties of Determinants

a.) Given \( A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix} \). Find \( A^3, -A, 3A \), and \( |A| \).

b.) \( A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 0 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix} \). Compute \( |A|, |B|, AB \) and \( |AB| \).

c.) \( A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 1 & 1 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \). Compute \( AB \) and \( |AB| \).

WUP6. Consider the list \{A, B, C, D, E, F\}.

a.) Show that interchanging \( B \) and \( F \) in this list requires an odd number of simple nearest neighbor interchanges.
\( \{A, B, C, D, E, F\} \rightarrow \{A, F, C, D, E, B\} \Rightarrow \) odd number of simple interchanges.

b.) Show that interchanging \( B \) and \( E \) in this list requires an odd number of simple nearest neighbor interchanges.
\( \{A, B, C, D, E, F\} \rightarrow \{A, E, C, D, B, F\} \) requires an odd number of simple interchanges.

Interchanging any two symbols in a list, while leaving the others in their original positions, requires an odd numbers of simple interchanges. Thus any interchange of two symbols in a list is an odd interchange.

WUP7. Use a brute force calculation to show that \( |A| |B| = AB | \) for the matrices below.
\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \]

**WUP8.** Use a brute force calculation to show that \(|A||B| = |AB|\) for the matrices below.

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \]

**WUP9.** Certain 2 \( \times \) 2 matrices have integer elements and determinants equal to 1. It is claimed that the inverses of these matrices also have integer elements. Show this for:

\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and for} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

Use the expression for the inverse of a matrix in terms of its classical adjoint.

**WUP10.** Consider a 2 \( \times \) 2 matrix and develop its inverse by solving the set of equations below.

\[ A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Repeat for:

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

Collect the terms that represent \(|A|\) and compare with the general expression for an inverse in terms of the classical adjoint.

**WUP11.** Consider the set of equations: \(ax + by = e\) and \(cx + dy = f\).

a.) Solve the equations using the addition/subtraction method. To start, multiply the first by \(d\) and the second by \(b\). Subtract the new second equation from the new first.

b.) Write the system of equation in matrix form.

c.) Solve the system of equations using Cramer’s rule.

d.) Compare.
Problems:

1.) Given the matrix \(A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\),

a.) Compute the determinant \(|A|\).

b.) Compute \(C\), the matrix of the cofactors of each element.

c.) Compute the classical adjoint: \(\text{adj} \ A = C^t\).

d.) Compute \(A^{-1} = \frac{(\text{adj} \ A)}{|A|}\).

2. a.) An application of matrix techniques to geometric optics requires calculating \(|A|\), the determinant of the matrix product below:

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & T_0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -D_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -D_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & T_7 \\ 0 & 1 \end{bmatrix}
\]

Find \(|A|\). Think before you calculate!

b.) Later in the calculation, the expression \(a_{21} - \frac{a_{11}a_{22}}{a_{12}}\) is to be simplified. Re-express this quantity in terms of only one of the \(a_{ij}\).

3. a.) Compute the determinants \(\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\) and \(\begin{vmatrix} 2 & -2 \\ 2 & 2 \end{vmatrix}\).

c.) Consider a 3 \(\times\) 3 matrix \(M\) for which \(|M| = 5\). What is \(|3M|\)?

The matrix \(3M\) has elements \([3M]_{ij} = 3m_{ij}\).

d.) For a scalar \(c\) and an \(n \times n\) matrix \(M\), find the ratio of \(|cM|\) to \(|M|\).

4. Consider the matrix \(B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\). Compute the inverse of \(B\) by finding:

the determinant of \(B\)

the nine cofactors (organize your work!)
5. Use the definition of matrix multiplication $$(A \cdot B)_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}$$ to establish that matrix multiplication is associative: $$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$. Use the fact that all the elements of the matrices are members of the same scalar field. The properties satisfied by elements of a field are listed in the Vector Spaces handout and may be found by searching for field at mathworld.wolfram.com.

6. Show that $1_A = A$ for all $A$ for which the multiplication is defined. Recall that an $$(n \times n) \otimes (n \times m) \rightarrow (n \times m)$$.

7. Show that $$(A \cdot B)^t = B^t \cdot A^t$$. Start with $$(A \cdot B)_{ij} = \sum_{\ell=1}^{n} a_{i\ell} b_{\ell j}$$. Express the $ij$ element of the transpose, express elements in the sum in terms of the elements of the transposed matrices and reassemble using the definition of the matrix product. As a preamble, show that if $A$ is an $n \times m$ and $B$ is an $m \times p$ so that matrix multiplication $A \cdot B$ is defined, then the product $B^t \cdot A^t$ will also have the correct dimensional arrangement to be defined.

8. Find a representation for $(A \cdot B)^{-1}$ in terms of $A^{-1}$ and $B^{-1}$. See: A right-side inverse is also a left-side inverse.

9. For the matrix $A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{bmatrix}$, $|A| = 45$. What is $|A^{-1}|$?

Show that $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector. Find a vector orthogonal to $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ that is also an eigenvector with the same eigenvalue. If an eigenvalue is $n$-fold degenerate (repeated $n$ times) then there is an $n$ dimensional subspace of eigenvectors with that eigenvalue. Choose an orthogonal basis set of the
subspace to represent the eigenvectors (and order them to follow the right-hand rule for 3D problems).

12.) Compute $e^A$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Begin by recalling the Taylor's series expansion of $e^x$ and computing $A^2, A^3$ and $A^4$. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$. Also find $e^B$ where $B = \begin{bmatrix} 0 & \theta \\ -\theta & 0 \end{bmatrix}$.

This prescription defines a function of a matrix $A$ for square matrices only, and the zero power of the matrix $A$ is interpreted to be the identity $1$.

13.) Show that the product of two $n \times n$ diagonal matrices is a diagonal matrix with elements that are the product of the corresponding elements.

14.) Use the properties of matrices to compute the determinant

$$\begin{vmatrix} a+1 & a+2 & a+3 \\ 2 & 4 & 6 \\ 7 & 7 & 7 \end{vmatrix}$$

Identify each property that you use to simplify the task.

15.) Compute the determinant $|A|$ directly and after subtracting a multiple of one row from another by expanding in minors of the then sparsely populated row. Repeat for $|B|$.

$$|A| = \begin{vmatrix} 4 & 5 & 11 \\ 8 & 11 & 22 \\ 3 & 4 & 1 \end{vmatrix} \quad |B| = \begin{vmatrix} 3 & 4 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{vmatrix} \quad (\text{Answers: } |A| = -29; \ |B| = 2)$$

16.) Show that $(\vec{A} \times \vec{B}) \cdot (\vec{A} \times \vec{B}) = A^2 \ B^2 - (\vec{A} \cdot \vec{B})^2 = A^2 \ B^2 \left[1 - \cos^2 \theta\right] = A^2 \ B^2 \left[\sin^2 \theta\right]$ and hence that $|\vec{A} \times \vec{B}| = AB \sin \theta$ paralleling $\vec{A} \cdot \vec{B} = AB \cos \theta$. The inner product gauges the degree to which vectors parallel and the cross product gauges the degree to which they are perpendicular.
17.) List seven properties of determinants.

18.) Given the matrix \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \),

a.) Compute the determinant \( |A| \) by expanding in terms of the cofactors of the first row.

b.) Compute the matrix of the cofactors of each element \( C \). Recall: \( \text{adj} \ A = C^t \).

c.) Compute its inverse \( A^{-1} = (\text{adj} A) / |A| \). Use the symbol \( D \) to represent \( |A| \). Write it all out as a large matrix with elements of the form \( (e_{ij} - f_{ij}) / D \).

d.) Under what condition does the algebraic matrix found in part \( d \) fail to provide an inverse for \( A \).

Partial Answer: 
\[ |A| = a(e_{ii} - f_{ii}) + b(f_{gi} - d_{gi}) + c(d_{hi} - e_{hi}) \]

19.) Two matrices are said to commute if \( A B = B A \). The commutator of the matrices is defined as \( [A, B] = A B - B A \). All matrices are \( n \times n \).
Show that: 
\( (A + B)(A - B) = A^2 - B^2 \) only if \( [A, B] = 0 \).

20.) The matrices \( A \) and \( B \) are diagonal. Show that \( A B = B A \) and hence \( [A, B] = 0 \). Assume that the diagonal elements of the two matrices have the values \{ \( a_1, a_2, a_3, \ldots a_n \) \} and \{ \( b_1, b_2, b_3, \ldots b_n \) \}.
Give a list of the diagonal elements of \( AB \). Give a list of the diagonal elements of \( A^2 \) and for \( A^m \) where \( m \) is an integer. Lesson Learned: Diagonal matrices commute.

21.) Check the identity: \( \sum_{i=1}^{3} \epsilon_{ijk} \epsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \). There are 81 distinct assignments for the indices \( \{ j, k, m, n \} \). Calculate both sides explicitly for the sets: \{ 1 2 1 2 \}, \{ 3 2 2 3 \}, \{ 1 2 3 1 \} and \{ 1 2 3 3 \}.

22.) Consider the permutation symbol \( \epsilon^{ijk}_{*} \). Give the values of: \( \epsilon^{**}_{ijk} \), \( \epsilon^{**}_{iji} \), \( \epsilon^{**}_{i*} \), \( \epsilon^{**}_{i**} \), and \( \epsilon^{**}_{**} \).

23.) a.) Compute the determinants of the matrices
b.) Compute the determinants of the transposes of those matrices.

24.) List nine special terms related to matrices.

25.) Expand the relation  
\[
\sum_{j,k=1}^{3} \epsilon_{2jk} A_j B_k = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{2jk} A_j B_k .
\]
Replace the subscripts using the substitutions 1 → x; 2 → y; 3 → z. Compare with \[ \bar{A} \times \bar{B} \] .

26.) Given two n x n matrices \( A \) and \( B \), does \( A \cdot B = 0 \) require that either \( A = 0 \) or that \( B = 0 \) where \( 0 \) is the n x n zero matrix?

27.) Given two n x n matrices \( A \) and \( B \), does \( |A \cdot B| = 0 \) require that \(|A| = 0 \) or that \(|B| = 0 \)? Explain.

28.) Express the following set of linear algebraic equations in matrix form and then solve them using Cramer's rule.

\[
\begin{align*}
x + y - z &= 6 \\
x - y + z &= 0 \\
x + y + 2z &= 0
\end{align*}
\]

29.) Under what condition does Cramer's rule fail to provide unique values for \( x, y \) and \( z \) given equations represented as:

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
3 \\
2
\end{bmatrix}
\]

30.) Determinants  
You must show your calculation steps, not just punch the calculator.
a.) Compute \[\begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}.\]

b. Use other than brute force to compute. *Minors and Cofactors* come to mind.

\[\begin{vmatrix} 2 & 7 & 1 \\ 1 & 7 & 2 \\ 0 & 2 & 0 \end{vmatrix}, \quad \begin{vmatrix} 2 & 7 & 1 \\ 1 & 7 & 2 \\ 1 & 3 & 1 \end{vmatrix}\]

c.) \[|A| = 4 ; |B| = 2.\] Give the values of each determinant below based on the values for the 3 x 3 matrices \(|A|\) and \(|B|\).

\[|AB| = |A^{-1}B| = 3B| = |A^t| = |B|\]

d.) Give the definition for the determinant of a 3 x 3 matrix with elements \(a_{mn}\) using summations and the permutation symbol \(\epsilon_{ijk}\).

31.) The volume of a parallelepiped with sides \(\{\vec{A}, \vec{B}, \vec{C}\}\) is \(\vec{A} \cdot (\vec{B} \times \vec{C})\). Show that this volume is can be computed as the determinant

\[\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}\]

32.) A general set of 3D coordinates are related to the Cartesian set by the transformation equations \(x(q_1,q_2,q_3), y(q_1,q_2,q_3), z(q_1,q_2,q_3)\). Small positive changes in each of the coordinates generate three small displacements characteristic of the general coordinates (assumed to be in RH order):

\[d\vec{r}_{q_1} = \frac{\partial x}{\partial q_1} dq_1 \hat{i} + \frac{\partial y}{\partial q_1} dq_1 \hat{j} + \frac{\partial z}{\partial q_1} dq_1 \hat{k}, \quad d\vec{r}_{q_2} = \frac{\partial x}{\partial q_2} dq_2 \hat{i} + \frac{\partial y}{\partial q_2} dq_2 \hat{j} + \frac{\partial z}{\partial q_2} dq_2 \hat{k}, \quad d\vec{r}_{q_3} = \frac{\partial x}{\partial q_3} dq_3 \hat{i} + \frac{\partial y}{\partial q_3} dq_3 \hat{j} + \frac{\partial z}{\partial q_3} dq_3 \hat{k}\]

and \(d\vec{r} = \left(d\vec{r}_{q_1} \times d\vec{r}_{q_2}\right) > 0\). Show that the volume element in the general system can be represented as:
\[
dV = \begin{vmatrix}
\frac{\partial x}{\partial q_1} dq_1 & \frac{\partial y}{\partial q_1} dq_1 & \frac{\partial z}{\partial q_1} dq_1 \\
\frac{\partial x}{\partial q_2} dq_2 & \frac{\partial y}{\partial q_2} dq_2 & \frac{\partial z}{\partial q_2} dq_2 \\
\frac{\partial x}{\partial q_3} dq_3 & \frac{\partial y}{\partial q_3} dq_3 & \frac{\partial z}{\partial q_3} dq_3
\end{vmatrix} dq_1 dq_2 dq_3
\]

where
\[
\begin{vmatrix}
\frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\
\frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2} \\
\frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3}
\end{vmatrix}
\]
is the determinant of the Jacobian matrix for the transformation.

33.) Compute \(dV = \begin{vmatrix}
\frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\
\frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi}
\end{vmatrix} \, dr \, d\theta \, d\phi\). It is helpful to factor \(r^2 \sin \theta\) out of each term in the expansion of the determinant. Refer to the previous problem.

34.) The identity \[\sum_{i=1}^{3} \epsilon_{ijk} \epsilon_{ilm} = \delta_{j\ell} \delta_{km} - \delta_{jm} \delta_{k\ell}\] is used in the proofs of vector identities.
Verify that the identity is valid for the following specific sets of index values by direct calculation of both sides.
\[{j, k, \ell, m} = \{1, 3, 1, 3\}; \{2, 1, 1, 2\}; \{1, 3, 2, 1\}; \{2, 2, 2, 1\}; \{2, 2, 2, 2\}\]

35.) The definition of the determinant of a 3 x 3 matrix \(\Delta\) is \[|\Delta| = \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_{ii} a_{jj} a_{kk}\]. Show that this is equivalent to \[|\Delta| = \left(\frac{1}{2!}\right) \sum_{i,m,n=1}^{3} \sum_{i,j,k=1}^{3} \epsilon_{imn} \epsilon_{ijk} a_{ii} a_{jj} a_{kk}\]. Argue effectively that for an \(n \times n\):
\[|\Delta| = \left(\frac{1}{n!}\right) \sum_{i_1,i_2,...,i_n=1}^{n} \sum_{j_1,j_2,...,j_n=1}^{n} \epsilon_{i_1j_1} \epsilon_{i_2j_2} ... \epsilon_{i_nj_n} a_{i_1j_1} a_{i_2j_2} ... a_{i_nj_n}\]

36.) Consider the velocity vector: \(\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}\). The identification of the 1 component as the \(x\) component and so on follows the scheme \(\{x, y, z\} \rightarrow \{1, 2, 3\}\). Consider the matrix \(\mathbf{W}\) formed with elements \(w_{ij} = v_i v_j\). Compute the trace of \(\mathbf{W}\). (\(\text{Tr}[\mathbf{W}] = w_{11} + w_{22} + w_{33}\)). Now
consider a change of coordinate system change to one rotated by $30^\circ$ about the $z$ axis. Which elements of $\mathbb{W}$ would change their values as viewed from this rotated frame. What would happen to the value $\text{Tr}[\mathbb{W}]$? The trace of a matrix is a scalar value. Rotations preserve the values of scalars in general and inner products in particular. Note that any two vectors $\vec{A}$ and $\vec{B}$ could be used to form a matrix $\mathbb{D}$ with similar results.

$$\mathbb{W} = \begin{bmatrix} v_xv_x & v_xv_y & v_xv_z \\ v_yv_x & v_yv_y & v_yv_z \\ v_zv_x & v_zv_y & v_zv_z \end{bmatrix}$$

$$\mathbb{D} = \begin{bmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{bmatrix}$$

37.) If $\mathbb{FLP}$ is the $n \times n$ identity matrix is modified by interchanging rows $k$ and $m$, left side multiplication of $A$ by $\mathbb{FLP}$ interchanges row $k$ and $m$ in $A$. Show that the determinant of $\mathbb{FLP}$ is -1. Verify that: $[\mathbb{FLP}]_{ij} = \delta_{ij} (1 - \delta_{im} - \delta_{kj}) + \delta_{im} \delta_{kj} + \delta_{ik} \delta_{mj}$.

38.) Consider the $n \times n$ matrices $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compare $AB$ and $BA$.

Matrices with the property that $AB = -BA$ are said to anti-commute.

39.) The trace of a matrix is defined to be the sum of its diagonal elements.

$$\text{Tr}[\mathbb{M}] = \sum_{ij} \delta_{ij} m_{ij}$$ (Assume that all matrices are $n \times n$.)

(a.) Show that $\text{Tr}[AB] = \text{Tr}[BA]$.

(b.) Show that $\text{Tr}[ABC] = \text{Tr}[CAB]$.

(c.) Show that $\text{Tr}[ABC]$ need not equal $\text{Tr}[BAC]$. See the problem above about matrices that do not commute. Choose $C$ to yield a product with a trace.

40.) Show that the product of two diagonal matrices is a diagonal matrix. Give the form of the diagonal elements of the product matrix. What is the determinant of a diagonal matrix? (the diagonal matrices are square $\Rightarrow n \times n$.)
41.) Show that: $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$. Conclude that $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \geq 0$$ for vectors in right-hand-rule order.

42.) Show that the identity below follows from the definition of determinant and the properties of the permutation symbol. Summations over repeated indices are understood.

$$D_{ijk...} = \varepsilon_{imn...} A_i A_j A_k ... = \varepsilon_{ijk...} |\mathbf{A}|$$

Use this representation to prove at least two properties of determinants.

43.) The identity below is established in the previous problem. Summations over repeated indices are understood.

$$D_{ijk...} = \varepsilon_{imn...} A_i A_j A_k ... = \varepsilon_{ijk...} |\mathbf{A}|$$

Use this representation to show that: $$|\mathbf{A}\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$$. To get started, note that:

$$D_{ijk...} = \varepsilon_{imn...} A_i B_j A_m B_n A_k B_n ... = \varepsilon_{ijk...} |\mathbf{A}\mathbf{B}|$$

44.) Show that $$\delta_{m+1,n} = \delta_{m,n-1}$$.

45.) Suppose the $$\mathbf{A}$$ is a square matrix with non-zero elements on the diagonal only. Let those elements be $$\lambda_m$$ for the row $$m$$ diagonal element. Show that $$(\mathbf{A})^k$$ is a diagonal matrix with elements $$\lambda_m^k$$ on the diagonal in row $$m$$ diagonal and zeroes elsewhere.

46.) Consider a set of the mutually ortho-normal directions listed in right-hand rule order $${\hat{e}_1, \hat{e}_2, \hat{e}_3}$$. Using problem 41 and the result that $$\varepsilon_{ijk} = \hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k)$$, it follows that:

$$\varepsilon_{ijk} = \begin{vmatrix} \ldots \hat{e}_i \ldots \\ \ldots \hat{e}_j \ldots \\ \ldots \hat{e}_k \ldots \end{vmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \hat{e}_i & \hat{e}_j & \hat{e}_k \end{bmatrix}$$

That is, the determinant of the matrix formed by using the components of any set of three vectors drawn from the three mutually orthogonal directions as the elements of the rows or columns can be represented by the permutation symbol. Any set means that any one of the three mutually orthogonal
directions can appear 0, 1, 2 or 3 times. The vectors can be slipped in as columns as well as rows because the determinant of the transpose of a matrix is the same as the determinant of the matrix.

a.) Use the results above to represent \( \varepsilon_{ijk} \varepsilon_{klm} \) using the row determinant and then the column determinant.

b.) Use the result that \( |A| |B| = |A \cdot B| \). Note that each element of \( A \cdot B \) is the inner product of two of the directions.

c.) Expand the resulting form of \( |A \cdot B| \) carefully and completely. Collect terms.

d.) Conclude that \( \varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \).

e.) Deduce an similar expression for \( \varepsilon_{kij} \varepsilon_{klm} \).

47.) The Lorentz transformations are:

\[
\beta = \frac{v}{c}; \quad \gamma = \left[1 - \beta^2\right]^{-\frac{1}{2}}
\]

\[
t' = \gamma \left[t - \beta \frac{1}{c} x\right] \quad x' = \gamma \left[x - \beta \frac{1}{c} t\right] \quad y' = y \quad z' = z
\]

It is assume that the primed and unprimed coordinate systems coincide at time \( t = t' = 0 \) and that the primed system has a velocity of \( v \) in the \( x \) direction relative to the unprimed.

a.) Rewrite these equations as equations for \( ct, x, y, \) and \( z \).

b.) Write your equations in matrix form.

\[
\begin{bmatrix}
ct \\
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
ct' \\
x' \\
y' \\
z
\end{bmatrix}
\]

i.e., in the form:

\[
\begin{bmatrix}
ct' \\
x'
\end{bmatrix} = \eta
\begin{bmatrix}
1 & \ldots & 1 \\
\ldots & \ldots & \ldots
\end{bmatrix}
\begin{bmatrix}
ct \\
x
\end{bmatrix}
\]

as you isolate groupings of interest. What are the off diagonal entries? What does \( \eta \) represent?

c.) Consider the transformation from unprimed to primed at \( v_1 \hat{i} \) followed by a transformation from the primed coordinates to double-primed coordinates using \( v_2 \hat{i} \). Find the matrix that represents the transformation from unprimed to double primed coordinates. You may suppress the \( y \) and \( z \) elements for parts c and d.

Hint: Use the reference form: \( \begin{bmatrix} ct' \\ x' \end{bmatrix} = \eta \begin{bmatrix} 1 & \ldots & 1 \\ \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \) as you isolate groupings of interest. What are the off diagonal entries? What does \( \eta \) represent?

d.) Use the result of the previous part to deduce the relativistic velocity addition rule for successive velocity transformations (boosts) in the \( x \) direction. Call the sum \( u \). Give \( u(\beta_1, \beta_2) \)

e.) Show that \( \gamma_u = \left[1 - (\frac{u}{c})^2\right]^{-\frac{1}{2}} = (1+\beta_1\beta_2) \gamma_1 \gamma_2 \).
48.) Prove that the determinant of a matrix block diagonal form is the product of the determinants of the blocks. Prove the proposition for a matrix with two diagonal blocks and then complete the proof using induction on the number of blocks. Alternatively, one can use: 

\[ \mathcal{E}_{j\cdots m, n\cdots i_q} = \mathcal{E}_{j\cdots m}^{12\cdots M} \mathcal{E}_{n\cdots i_q}^{M+1\cdots N} \]

for all cases that multiply a non-zero product of elements and the definition of the determinant of an \(N \times N\) matrix in the case of one with \(M \times M\) and \((N - M) \times (N - M)\) blocks.

49.) The determinant of a matrix block diagonal form is the product of the determinants of the blocks. Evaluate the following determinant using expansion by minors of the first row. Compare the result with the product of the determinants of the two diagonal blocks.

\[
\begin{vmatrix}
1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 4 & 3 & 0 \\
0 & 3 & 2 & 0 \\
\end{vmatrix}
= 
\begin{vmatrix}
1 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 \\
0 & 0 & 4 & 3 \\
0 & 0 & 3 & 2 \\
\end{vmatrix}
\]

50.) Consider a general 2 by 2 matrix \([a \ b] [c \ d]\). Find its inverse as the matrix of cofactors divided by the determinant. Find the transpose of the matrix. Find the inverse of the transpose. Comment on the results. Recall that the determinant of the transpose is equal to the determinant of the original square matrix. Argue that, for square matrices, the transpose of the inverse is the inverse of the transpose. Search the web. Is the transpose of the inverse equal to the inverse of the transpose?

51.) Consider the matrices \([1 \ 0] [0 \ 2]\) and \([7 \ -1] [-1 \ 5]\). Is the product of symmetric matrices necessarily a symmetric matrix?

52.) Real symmetric matrices can represent self-adjoint operators over a real inner product space. Show that if \(A, B\) and \(AB\) are symmetric, then \(AB = BA\). That is: the matrices commute.
References:


5. Mathworld.wolfram.com


7. This handout is based on lectures given by Professor Uldrich at Rice University in 1966. He in turn referenced the writings in Linear Algebra by Leonard E. Dickson.