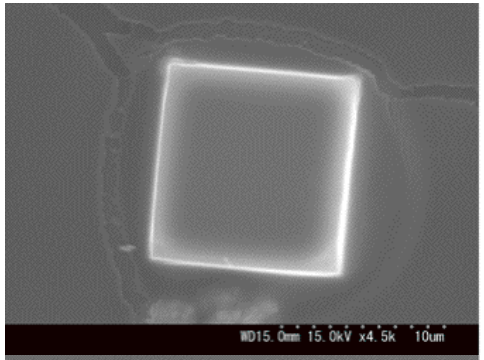
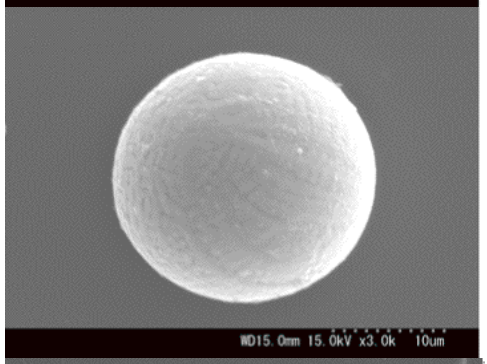
Airborne particles

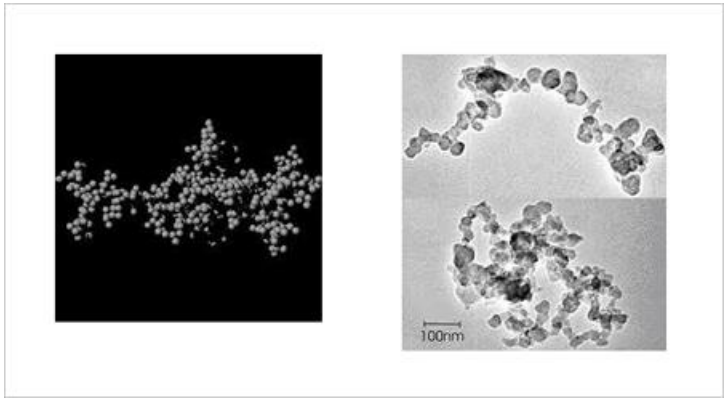


A cubic sea-salt crystal

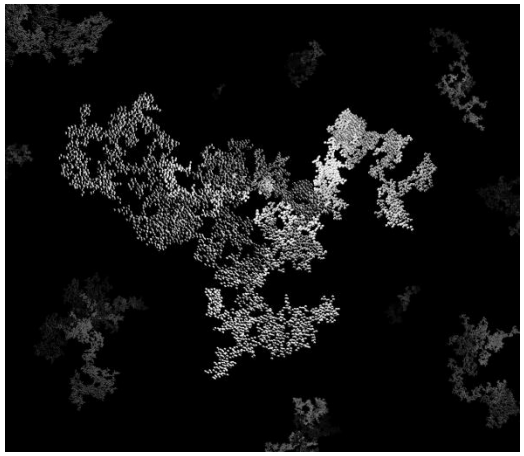


Surface pattern of one of cosmic dust particles

A particle containing CaSO4 crystals



Structure profile of many irregular airborne particles



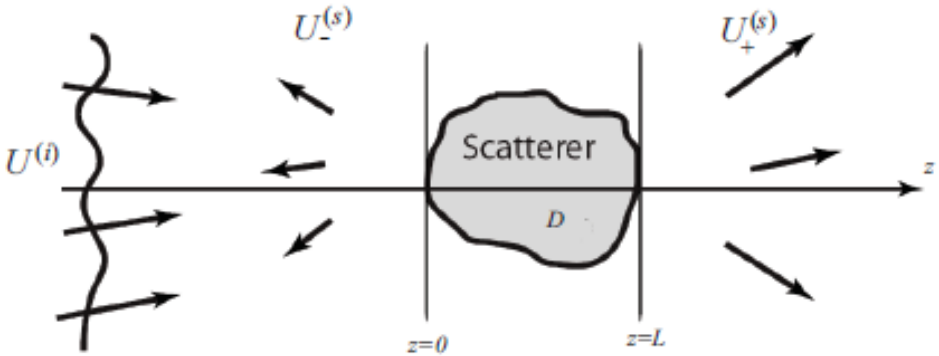
Fractal structure of soot particles in the air

http://chem.sci.utsunomiya-u.ac.jp/v10n2/SSato/SSato_body.html

https://news.slac.stanford.edu/sites/default/files/press-release/slac-pr-lcls-soot.pdf

https://www.wolframscience.com/summerschool/2013/alumni/subedi.html

Born series



$$U^{(t)}(\mathbf{r};\omega) = U^{(i)}(\mathbf{r};\omega) + U_{\pm}^{(s)}(\mathbf{r};\omega),$$

$$\nabla^2 U^{(t)}(\mathbf{r};\omega) + k^2 n^2(\mathbf{r};\omega) U^{(t)}(\mathbf{r};\omega) = 0,$$

$$\nabla^2 U^{(t)}(\mathbf{r};\omega) + k^2 U^{(t)}(\mathbf{r};\omega) = -4\pi F(\mathbf{r};\omega) U^{(t)}(\mathbf{r};\omega),$$

$$F(\mathbf{r};\omega) = \begin{cases} \frac{k^2}{4\pi}[n^2(\mathbf{r};\omega) - 1], & \mathbf{r} \in D, \\ 0, & \text{otherwise}, \end{cases}$$

$$U^{(t)}(\mathbf{r};\omega) = U^{(i)}(\mathbf{r};\omega) + \int_D F(\mathbf{r}';\omega) U^{(t)}(\mathbf{r}';\omega) G(\mathbf{r}-\mathbf{r}';\omega) d^3r',$$

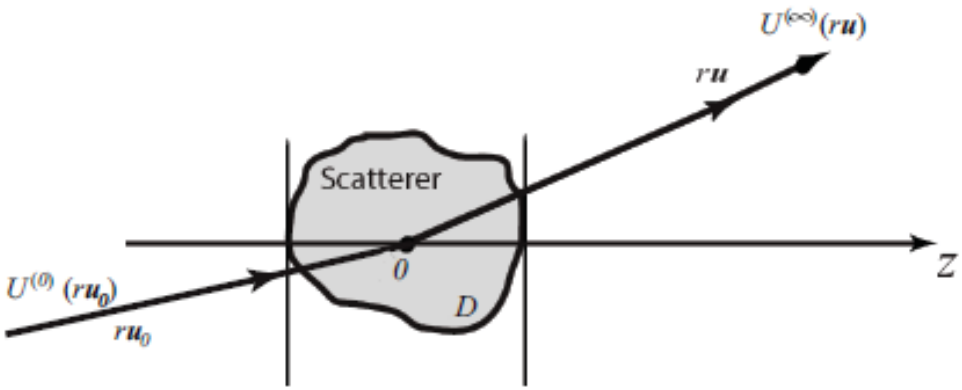
$$G(\mathbf{r}-\mathbf{r}';\omega) = \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}$$

First Born approximation

$$|U^{(s)}| << |U^{(i)}|$$

$$U^{(t)}(\mathbf{r};\omega) = U^{(i)}(\mathbf{r};\omega) + \int_D F(\mathbf{r}';\omega) U^{(i)}(\mathbf{r}';\omega) G(\mathbf{r}-\mathbf{r}';\omega) d^3r'$$

Far zone approximation



$$G(\mathbf{r}-\mathbf{r}';\omega) = \frac{\exp(ikr)}{r}\exp(-ik\mathbf{u}\cdot\mathbf{r}')$$

$$U^{(t)}(\mathbf{r};\omega) = U^{(i)}(\mathbf{r};\omega) + \frac{\exp(ikr)}{r}\int_D F(\mathbf{r}';\omega)U^{(i)}(\mathbf{r}';\omega)\exp(-ik\mathbf{u}\cdot\mathbf{r}')d^3r'.$$

$$U^{(i)}(\mathbf{r};\omega) = a^{(i)}(\omega)\exp(ik\mathbf{u}_0\cdot\mathbf{r})$$

$$U^{(t)}(\mathbf{r};\omega) = a^{(i)}(\omega)\left[\exp(ik\mathbf{u}_0\cdot\mathbf{r}) + \frac{\exp(ikr)}{r}\tilde{F}(k(\mathbf{u}-\mathbf{u}_0);\omega)\right]$$

$$\mathbf{K} = k(\mathbf{u}-\mathbf{u}_0) \qquad \widetilde{F}(\mathbf{K}) = \int_D F(\mathbf{r}';\omega)\exp\left[-i\mathbf{K}\cdot\mathbf{r}'\right]d^3r'$$

Scattering from sphere

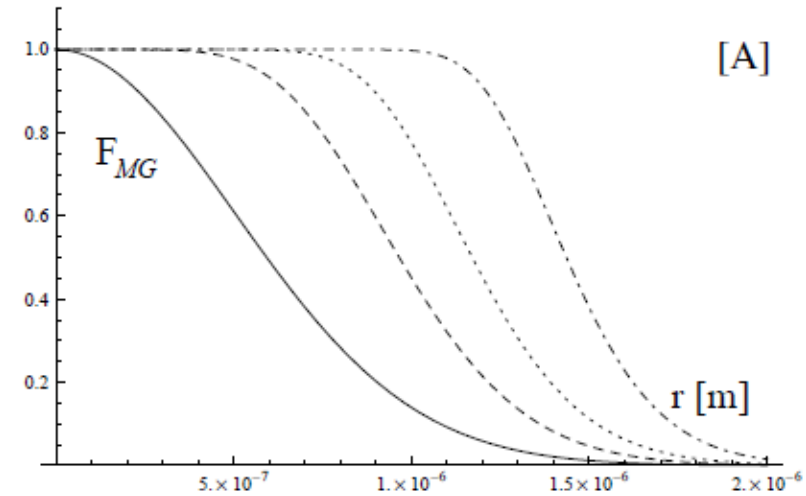
$$F^{(G)}(\mathbf{r};\omega) = \exp\left[-\frac{x^2+y^2+z^2}{2\sigma^2}\right]$$

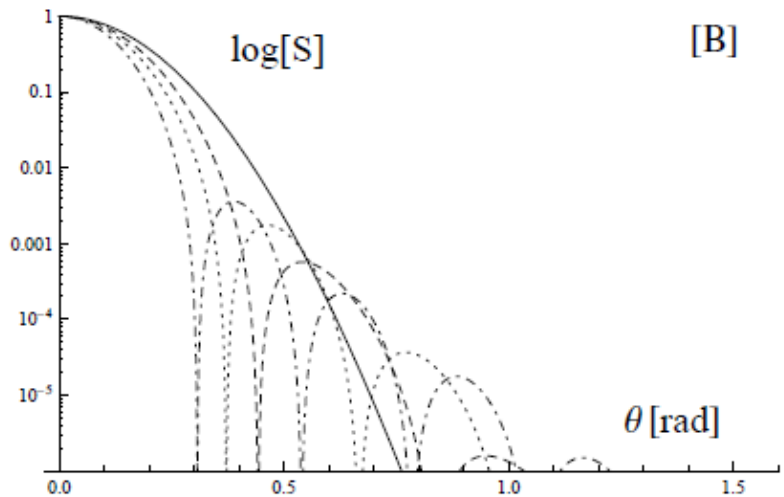
$$F^{(H)}(\mathbf{r};\omega) = \begin{cases} 1, & b \le r, \\ 0, & otherwise \end{cases}$$

$$F^{(MG)}(\mathbf{r};\omega) = \frac{1}{C_l}\sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l} \times \exp\left[-l\frac{x^2+y^2+z^2}{2\sigma^2}\right]$$

$$C_l = \sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l}$$

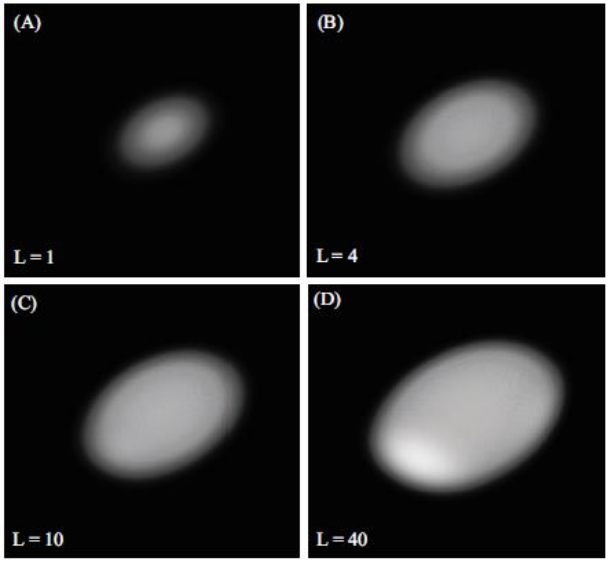
$$S^{(s)}_{MG}(r\mathbf{s};\omega) = \frac{S^{(i)}(\omega)(2\pi)^3\sigma^3}{r^2 C_l} \times \left(\sum_{l=1}^{L}\frac{(-1)^{l-1}}{l^{3/2}}\binom{L}{l}\exp\left\{\frac{-k^2}{2l}\sigma^2\left[(\mathrm{s}_x-\mathrm{s}_{0x})^2 + (\mathrm{s}_y-\mathrm{s}_{0y})^2 + (\mathrm{s}_z-\mathrm{s}_{0z})^2\right]\right\}\right)^2.$$





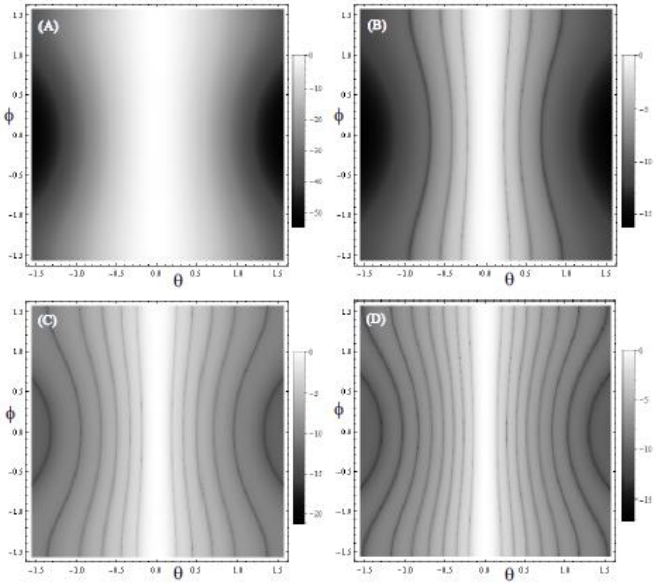
Scattering from semi-hard ellipsoids

$$F_E(\mathrm{r};\omega) = \frac{1}{C_l}\sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l} \times \exp\left[-l\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}+\frac{z^2}{2\sigma_z^2}\right)\right]$$



$$S_E^{(s)}(r\mathrm{s};\omega) = \frac{S^{(i)}(\omega)(2\pi)^3\sigma_x\sigma_y\sigma_z}{r^2 C_l} \times \left(\sum_{l=1}^{L}\frac{(-1)^{l-1}}{l^{3/2}}\binom{L}{l}\exp\left\{\frac{-k^2}{2l}\left[\sigma_x^2(\mathrm{s}_x-\mathrm{s}_{0x})^2 + \sigma_y^2(\mathrm{s}_y-\mathrm{s}_{0y})^2+\sigma_z^2(\mathrm{s}_z-\mathrm{s}_{0z})^2\right]\right\}\right)^2.$$

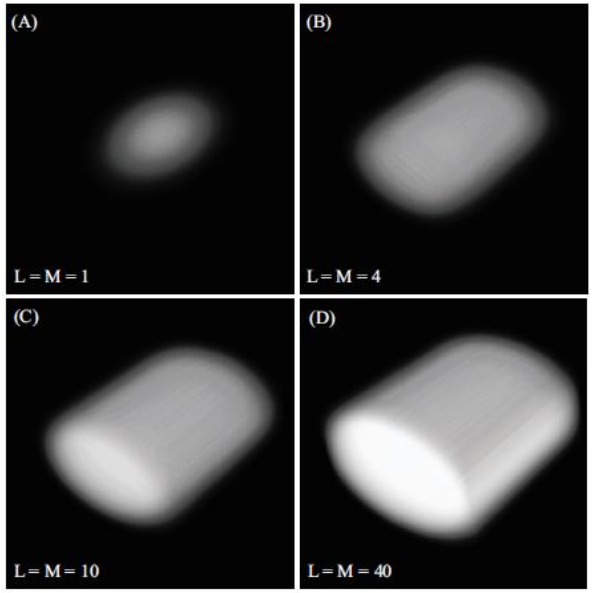
$$s_x = \sin\theta\cos\phi$$
$$s_y = \sin\theta\sin\phi$$
$$s_z = \cos\theta$$

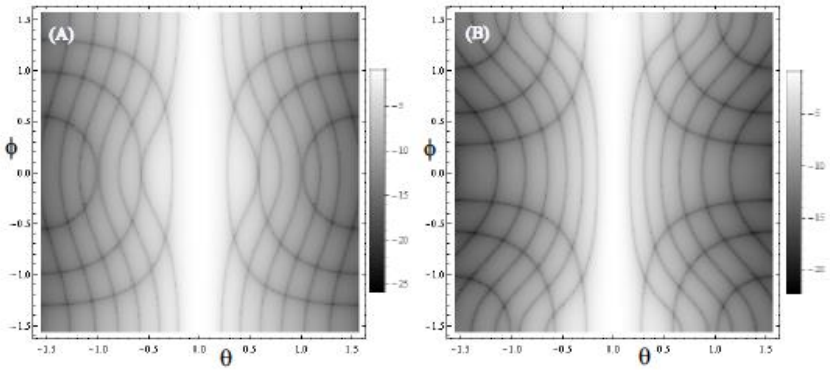


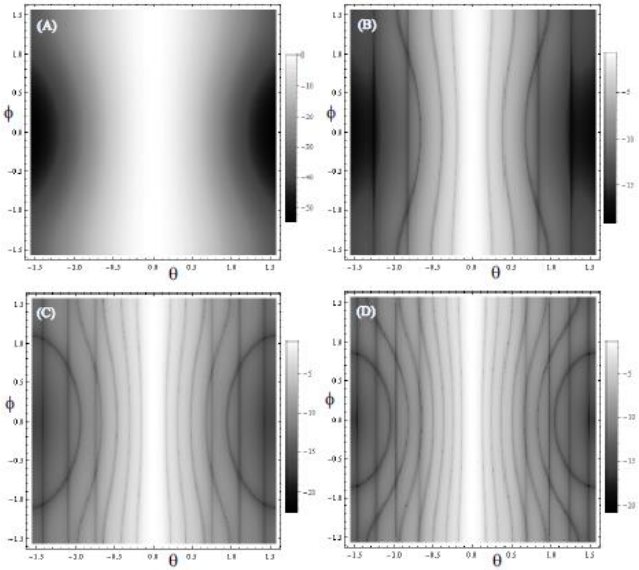
Scattering from semi-hard cylinders

$$F_C(\mathrm{r};\omega) = \frac{1}{C_l}\sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l}\exp\left[-l\left(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right)\right]$$
$$\times\frac{1}{C_m}\sum_{m=1}^{M}(-1)^{m-1}\binom{M}{m}\exp\left[-m\frac{z^2}{2\sigma_z^2}\right],$$

$$S_C^{(s)}(r\mathrm{s};\omega) = \frac{S^{(i)}(\omega)(2\pi)^3\sigma_x\sigma_y\sigma_z}{r^2 C_l C_n}\left\{\sum_{l=1}^{L}\frac{(-1)^{l-1}}{l}\binom{L}{l}\right.$$
$$\times\exp\left\{\frac{-k^2}{2l}\left[\sigma_x^2(\mathrm{s}_x-\mathrm{s}_{0x})^2+\sigma_y^2(\mathrm{s}_y-\mathrm{s}_{0y})^2\right]\right\}$$
$$\left.\times\sum_{m=1}^{M}\frac{(-1)^{m-1}}{\sqrt{m}}\binom{M}{m}\exp\left[\frac{-k^2\sigma_z^2}{2m}(\mathrm{s}_z-\mathrm{s}_{0z})^2\right]\right\}^2.$$

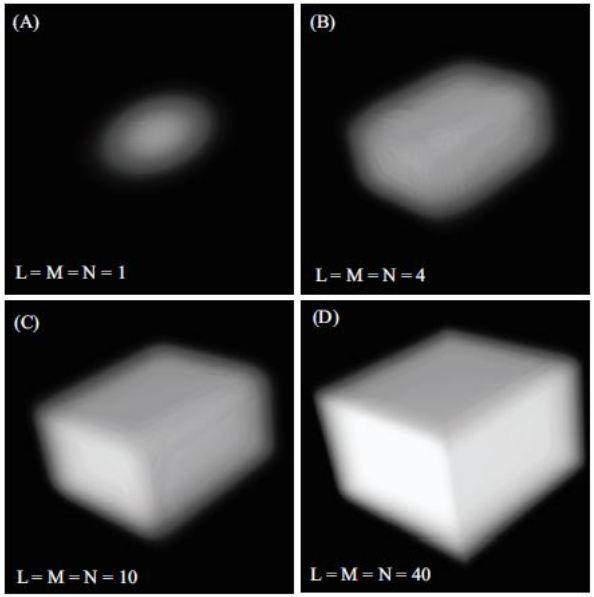




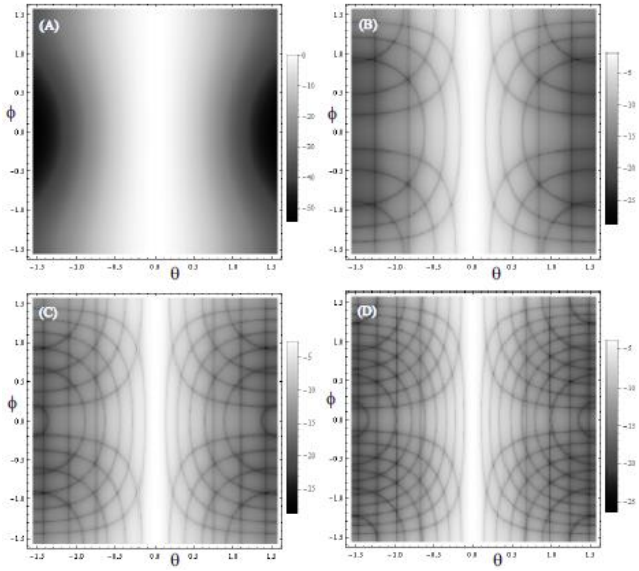


Scattering from semi-hard parallelepipeds

$$F_E(\mathrm{r};\omega) = \frac{1}{C_n}\sum_{n=1}^{N}(-1)^{n-1}\binom{N}{n}\exp\left[-n\frac{x^2}{2\sigma_x^2}\right]$$
$$\times\frac{1}{C_p}\sum_{p=1}^{P}(-1)^{p-1}\binom{P}{p}\exp\left[-p\frac{y^2}{2\sigma_y^2}\right]$$
$$\times\frac{1}{C_m}\sum_{m=1}^{M}(-1)^{m-1}\binom{M}{m}\exp\left[-m\frac{z^2}{2\sigma_z^2}\right]$$



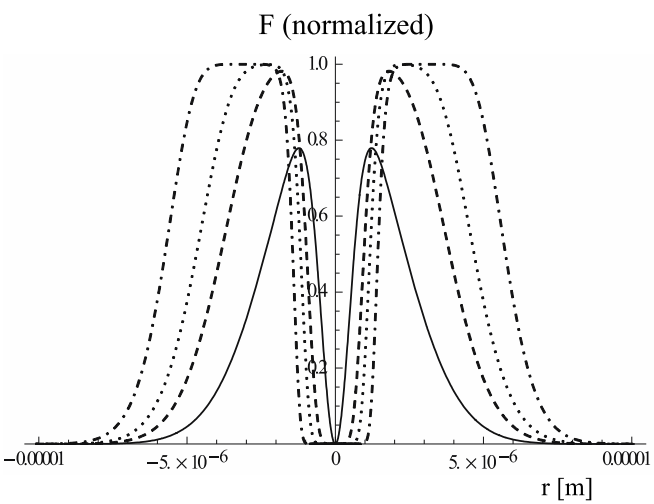
$$S_P^{(s)}(r\mathrm{s};\omega) = \frac{S^{(i)}(\omega)(2\pi)^3\sigma_x\sigma_y\sigma_z}{r^2C_nC_pC_m}$$
$$\times\left\{\sum_{n=1}^{N}\frac{(-1)^{n-1}}{\sqrt{n}}\binom{N}{n}\exp\left[\frac{-k^2\sigma_x^2}{2n}(\mathrm{s}_x-\mathrm{s}_{0x})^2\right]\right.$$
$$\times\sum_{p=1}^{P}\frac{(-1)^{p-1}}{\sqrt{p}}\binom{P}{p}\exp\left[\frac{-k^2\sigma_y^2}{2p}(\mathrm{s}_y-\mathrm{s}_{0y})^2\right]$$
$$\left.\times\sum_{m=1}^{M}\frac{(-1)^{m-1}}{\sqrt{m}}\binom{M}{m}\exp\left[\frac{-k^2\sigma_z^2}{2m}(\mathrm{s}_z-\mathrm{s}_{0z})^2\right]\right\}^2$$

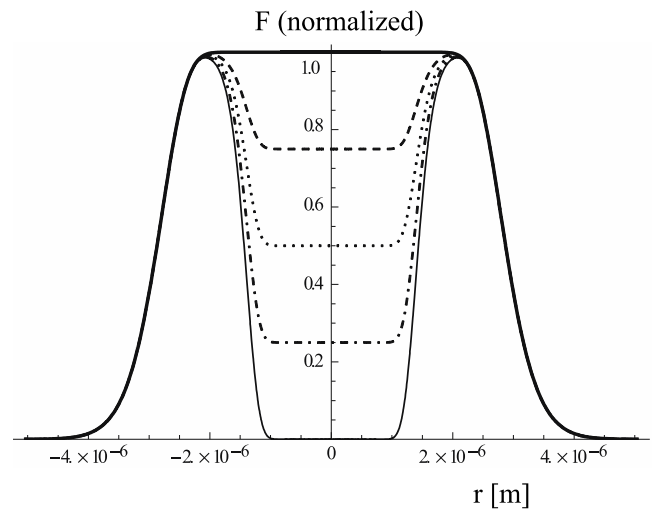


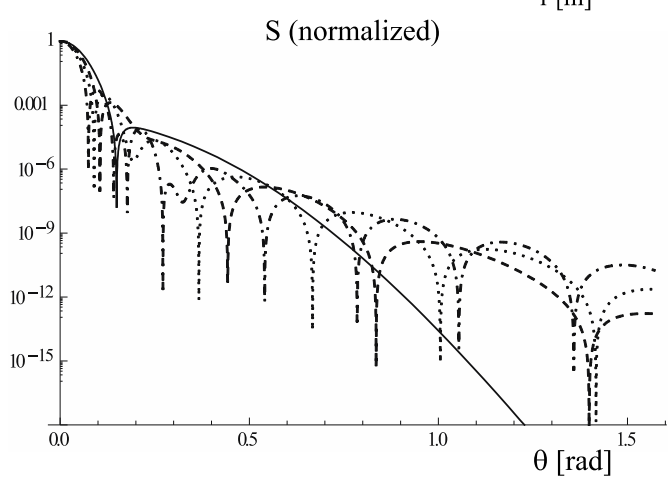
Scattering from hollow spheres

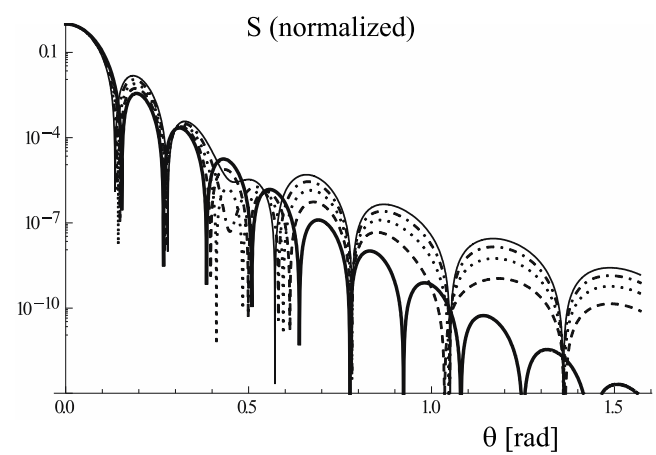
$$F_{HS}(\mathbf{r};\omega) = \frac{1}{C_l}\sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l} \times \left[h_o \exp\left[-l\frac{x^2+y^2+z^2}{2\sigma_o^2}\right] - h_i \exp\left[-l\frac{x^2+y^2+z^2}{2\sigma_i^2}\right]\right]$$

$$C_l = \sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l}$$

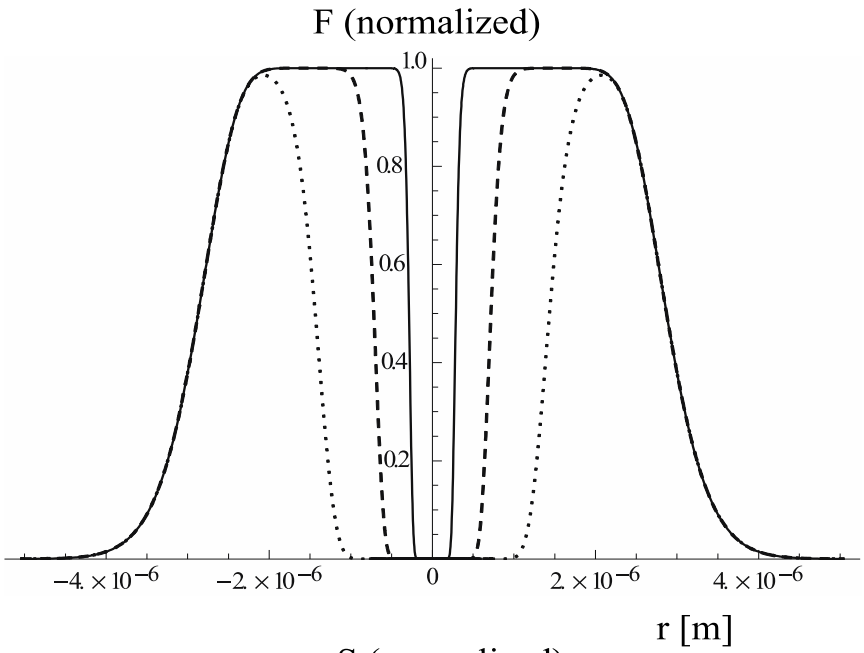


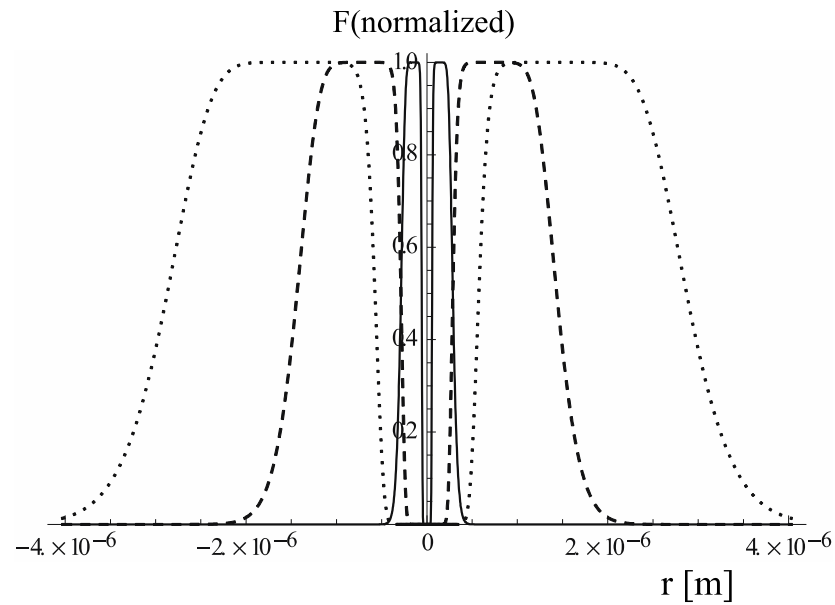


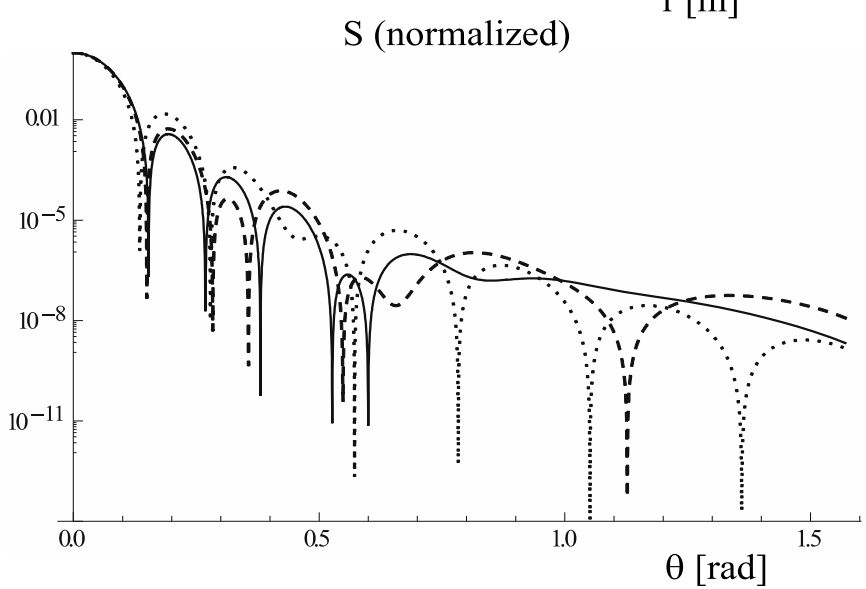


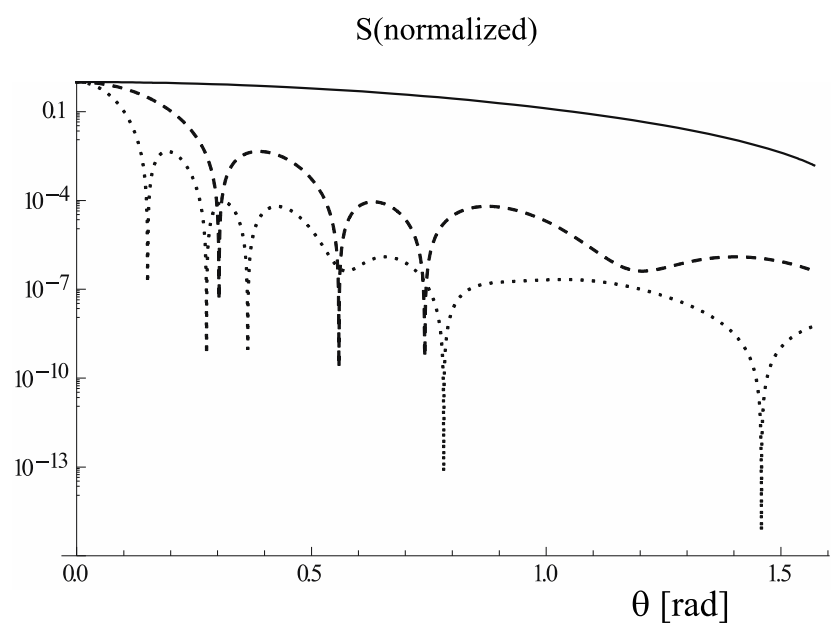


Scattering from hollow spheres









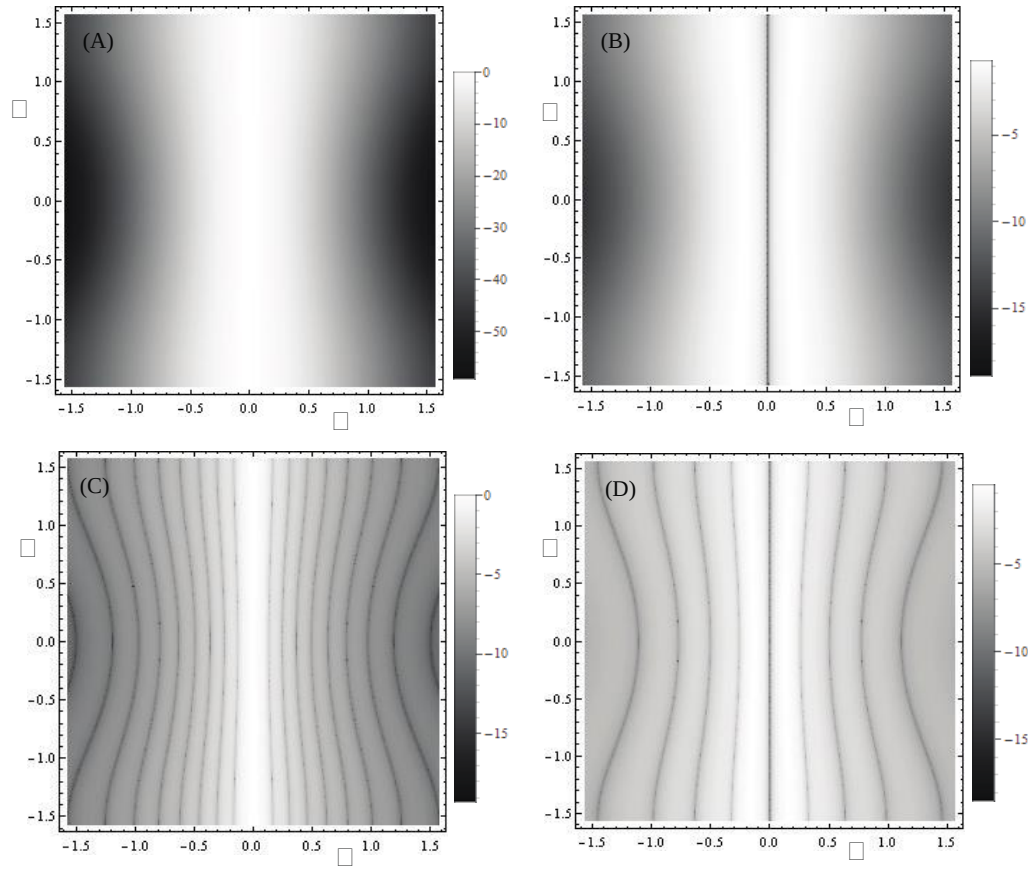
Scattering from hollow ellipsoids

$$F_{HE}(\mathbf{r};\omega) = \frac{1}{C_l}\sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l}$$

$$\times\left\{h_0 \exp\left[-l\left(\frac{x^2}{2\sigma_{xo}^2}+\frac{y^2}{2\sigma_{yo}^2}+\frac{z^2}{2\sigma_{zo}^2}\right)\right]\right.$$

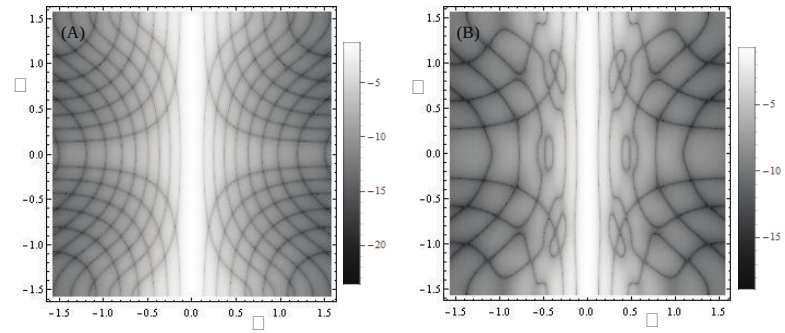
$$\left.-\,h_i \exp\left[-l\left(\frac{x^2}{2\sigma_{xo}^2}+\frac{y^2}{2\sigma_{yo}^2}+\frac{z^2}{2\sigma_{zo}^2}\right)\right]\right\}$$

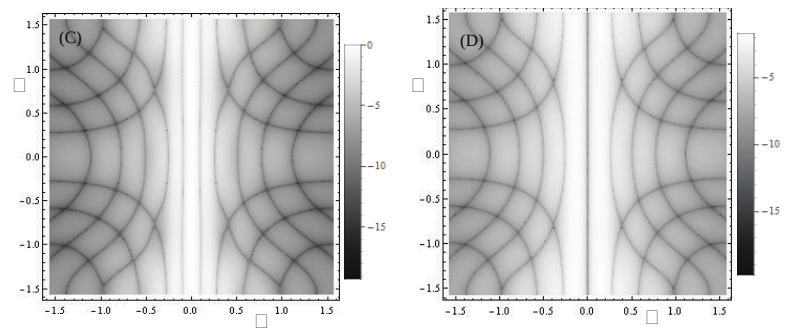
$$C_l = \sum_{l=1}^{L}(-1)^{l-1}\binom{L}{l}$$

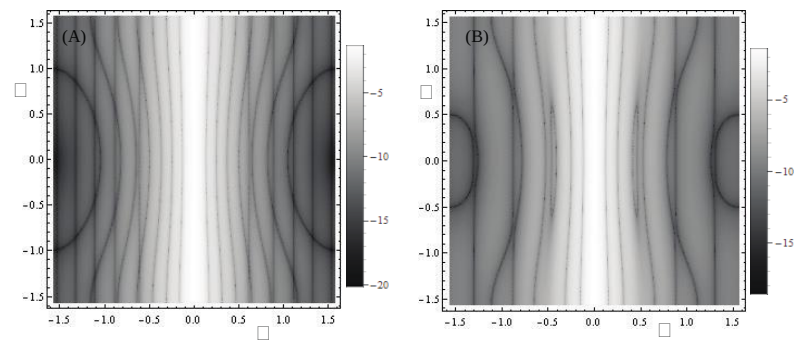


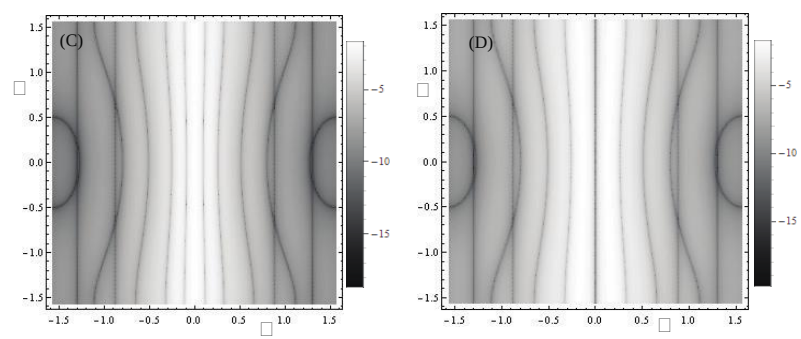
Scattering from hollow cylinders

$$F_{HC}(\mathbf{r};\omega) = h_0 \frac{1}{C_l} \sum_{l=1}^{L} (-1)^{l-1} \binom{L}{l}$$
$$\times \exp\left[-l\left(\frac{x^2}{2\sigma_{xo}^2} + \frac{y^2}{2\sigma_{yo}^2}\right)\right]$$
$$\times \frac{1}{C_m} \sum_{m=1}^{M} (-1)^{m-1} \binom{M}{m} \exp\left[-m\frac{z^2}{2\sigma_{zo}^2}\right]$$
$$- h_i \frac{1}{C_l} \sum_{l=1}^{L} (-1)^{l-1} \binom{L}{l} \exp\left[-l\left(\frac{x^2}{2\sigma_{xi}^2} + \frac{y^2}{2\sigma_{yi}^2}\right)\right]$$
$$\times \frac{1}{C_m} \sum_{m=1}^{M} (-1)^{m-1} \binom{M}{m} \exp\left[-m\frac{z^2}{2\sigma_{zi}^2}\right].$$



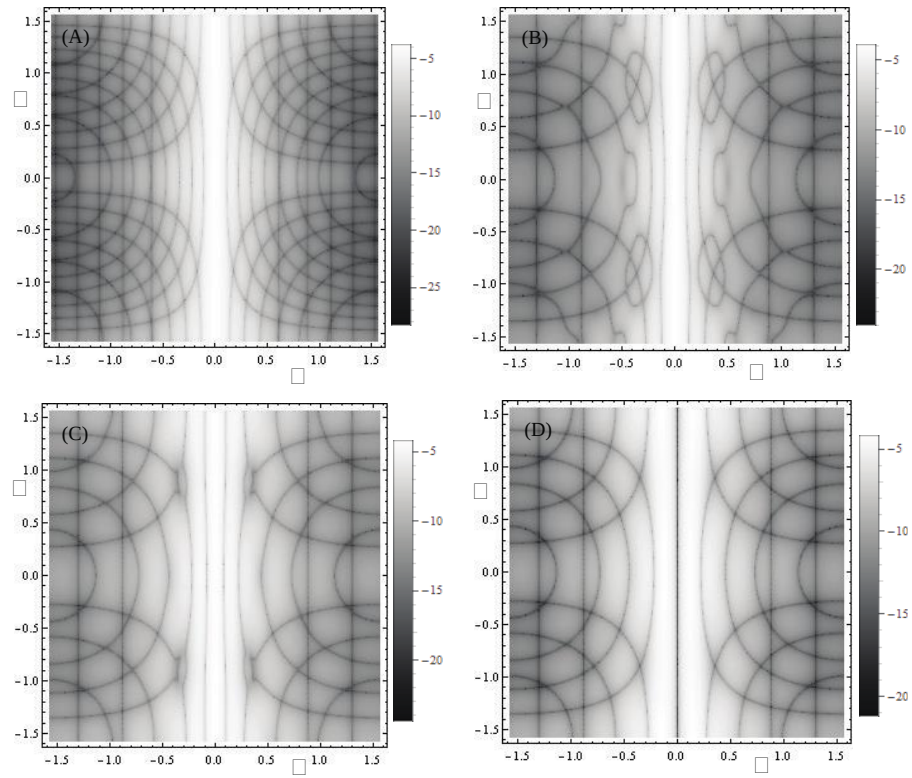


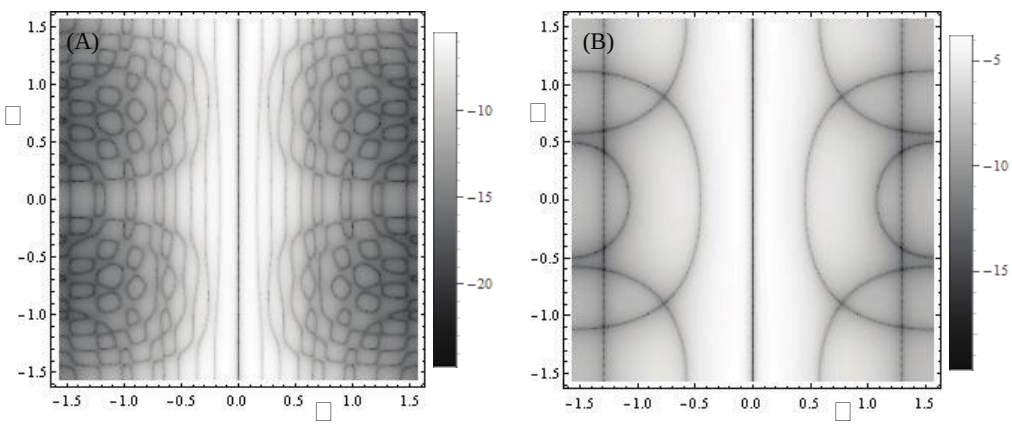




Scattering from hollow parallelepipeds

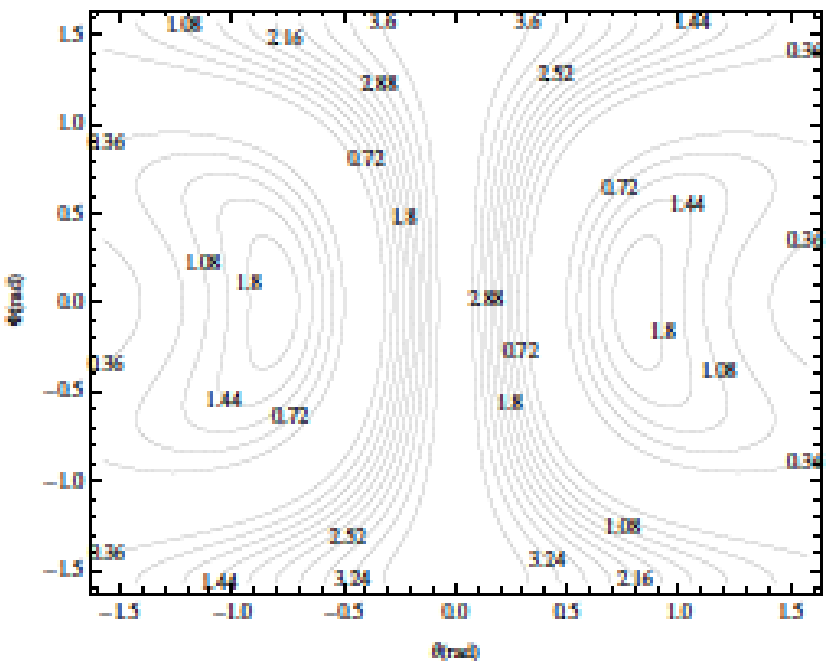
$$
\begin{aligned}
F_{HP}(\mathbf{r};\omega) = h_0 \frac{1}{C_n} \sum_{n=1}^{N} (-1)^{n-1} \binom{N}{n} \exp\left[-n \frac{x^2}{2\sigma_{xo}^2}\right] \\
\times \frac{1}{C_p} \sum_{p=1}^{P} (-1)^{p-1} \binom{P}{p} \exp\left[-p \frac{y^2}{2\sigma_{yo}^2}\right] \\
\times \frac{1}{C_m} \sum_{m=1}^{M} (-1)^{m-1} \binom{M}{m} \exp\left[-m \frac{z^2}{2\sigma_{zo}^2}\right] \\
- h_i \frac{1}{C_n} \sum_{n=1}^{N} (-1)^{n-1} \binom{N}{n} \exp\left[-n \frac{x^2}{2\sigma_{xo}^2}\right] \\
\times \frac{1}{C_p} \sum_{p=1}^{P} (-1)^{p-1} \binom{P}{p} \exp\left[-p \frac{y^2}{2\sigma_{yo}^2}\right] \\
\times \frac{1}{C_m} \sum_{m=1}^{M} (-1)^{m-1} \binom{M}{m} \exp\left[-m \frac{z^2}{2\sigma_{zo}^2}\right]
\end{aligned}
$$

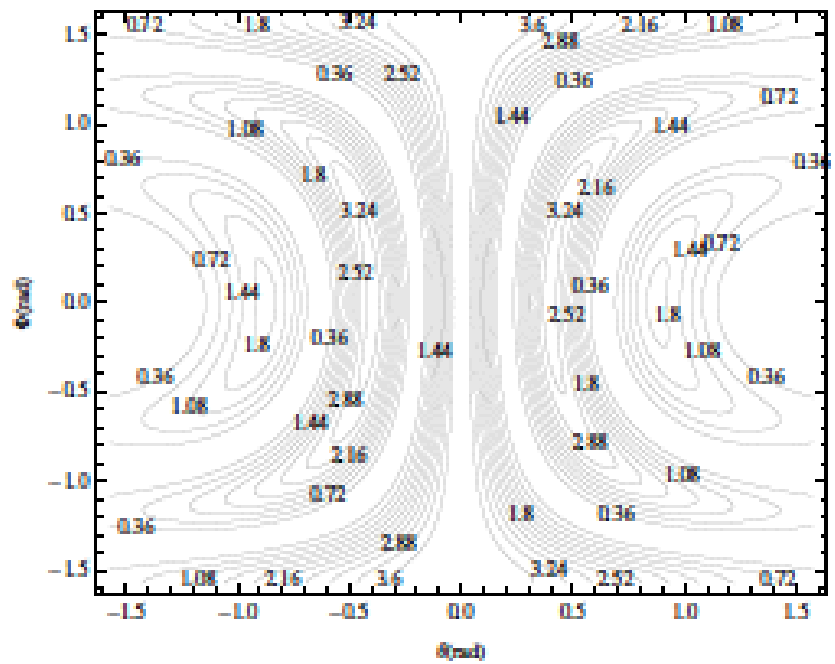




Scattering from two spheres: interference

$$U^{(s)}(\mathbf{r};\omega) = \frac{\sigma_s^3}{r}\exp\left[-\frac{\sigma_s^2 k^2}{2}[u_x^2 + u_y^2 + (u_z - u_{0z})^2]\right]$$
$$\times\left[\exp[-ik(x_c(u_x - u_{0x}) + z_c(u_z - u_{0z}))]\right.$$
$$\left.+\exp[-ik(-x_c(u_x - u_{0x}) + z_c(u_z - u_{0z}))]\right]$$





Arbitrary incident field

In the form of angular spectrum representation of plane waves

$$U(\boldsymbol{r},\omega)=\int a(\boldsymbol{s},\omega)\exp(ik\boldsymbol{s}\cdot\boldsymbol{r})d^2\boldsymbol{s}_\perp$$

$\boldsymbol{s}=\left(s_x,s_y,s_z\right)$ is a unit vector, $a(\boldsymbol{s},\omega)$ is the amplitude of a plane wave

$$a(\boldsymbol{s},\omega)=\left(\frac{k}{2\pi}\right)^2\int U^{(i)}(\boldsymbol{\rho}',\omega)\exp(-ik\boldsymbol{s}\cdot\boldsymbol{\rho}')d^2\rho'$$

Incident random field

$$W^{(i)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\langle U^{(i)*}(\mathbf{r}_1;\omega)U^{(i)}(\mathbf{r}_2;\omega)\rangle \qquad A^{(i)}(\mathbf{u}_1,\mathbf{u}_2;\omega)=\langle a^{(i)*}(\mathbf{u}_1;\omega)a^{(i)}(\mathbf{u}_2;\omega)\rangle$$

Example. Two partially correlated polychromatic plane waves:

$$a^{(i)}(\mathbf{u}',\omega)=a^{(i)}(\mathbf{u}_1',\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_1')+a^{(i)}(\mathbf{u}_2',\omega)\delta^{(2)}(\mathbf{u}'-\mathbf{u}_2')$$

$$a(\mathbf{u}_p',\mathbf{u}_q';\omega)=a_{pq}e^{-\frac{k^2\Delta^2}{2}(\mathbf{u}_q'-\mathbf{u}_p')^2}$$

$$\begin{aligned}A^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)=&\\ &a(\mathbf{u}_1',\mathbf{u}_1';\omega)\delta^2(\mathbf{u}'-\mathbf{u}_1')\delta^2(\mathbf{u}'-\mathbf{u}_1')\\ &+a(\mathbf{u}_2',\mathbf{u}_2';\omega)\delta^2(\mathbf{u}'-\mathbf{u}_2')\delta^2(\mathbf{u}'-\mathbf{u}_2')\\ &+a(\mathbf{u}_1',\mathbf{u}_2';\omega)\delta^2(\mathbf{u}'-\mathbf{u}_1')\delta^2(\mathbf{u}'-\mathbf{u}_2')\\ &+a(\mathbf{u}_2',\mathbf{u}_1';\omega)\delta^2(\mathbf{u}'-\mathbf{u}_2')\delta^2(\mathbf{u}'-\mathbf{u}_1')\end{aligned}$$

Δ is typical correlation width of two plane waves

$$W^{(t)}(\mathbf{r}_1,\mathbf{r}_2;\omega)=\iiiint M(\mathbf{u}_1,\mathbf{u}_2;\mathbf{u}_1',\mathbf{u}_2';\omega)A^{(i)}(\mathbf{u}_1',\mathbf{u}_2';\omega)e^{ik(\mathbf{u}_2.\mathbf{r}_2-\mathbf{u}_1.\mathbf{r}_1)}d^2\mathbf{u}_{1\perp}d^2\mathbf{u}_{2\perp}d^2\mathbf{u}_{1\perp}'d^2\mathbf{u}_{2\perp}'$$

Scattering from random collections

- Potential is deterministic function
- Positions are random

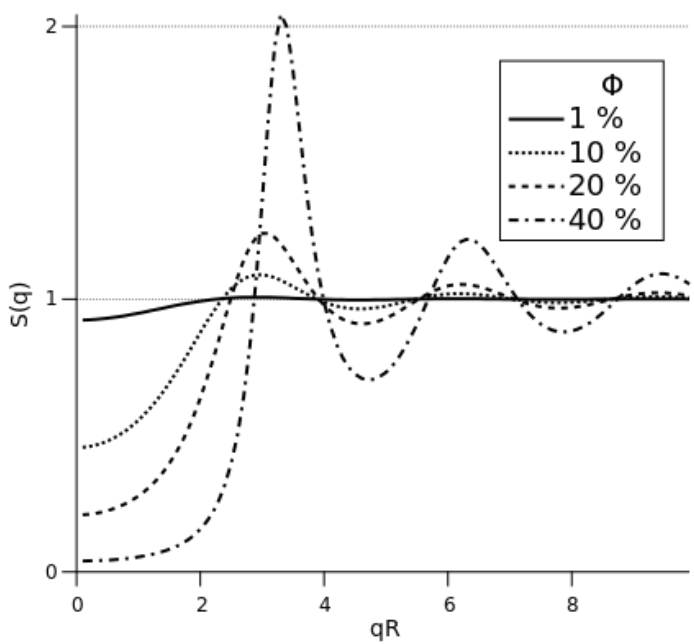
Pair-scattering matrix

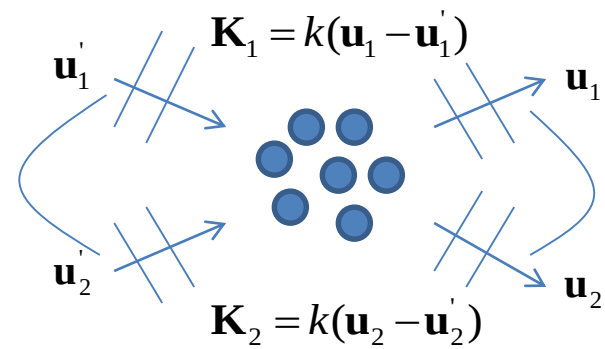
$$M(\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_1^{'},\mathbf{u}_2^{'}.\omega) = Q(-\mathbf{K}_1,\mathbf{K}_2,\omega)\iint f^*(\mathbf{R}_1,\omega)\, f(\mathbf{R}_2,\omega) e^{-i[\mathbf{K}_2.\mathbf{R}_2-\mathbf{K}_1.\mathbf{R}_1]} d^3\mathbf{R}_1 d^3\mathbf{R}_2$$

Pair-scattering factor $\quad Q(-\mathbf{K}_1,\mathbf{K}_2,\omega) = \langle \sum_m \sum_n e^{-i[\mathbf{K}_2.\mathbf{r}_n-\mathbf{K}_1.\mathbf{r}_m]} \rangle_{rm}$

$$Q(-\mathbf{K}_1,\mathbf{K}_2,\omega) = \sqrt{S(\mathbf{K}_1)}\sqrt{S(\mathbf{K}_2)}\eta(\mathbf{K}_2-\mathbf{K}_1)$$

Structure factor:
Percus-Yevick approximation

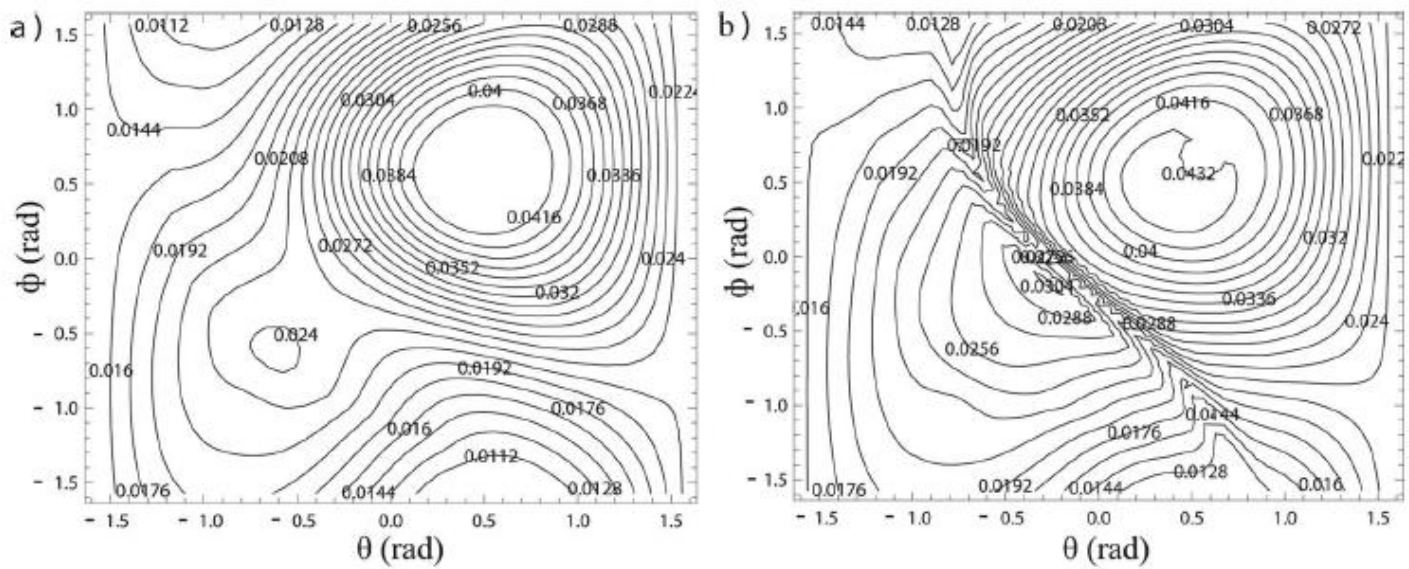


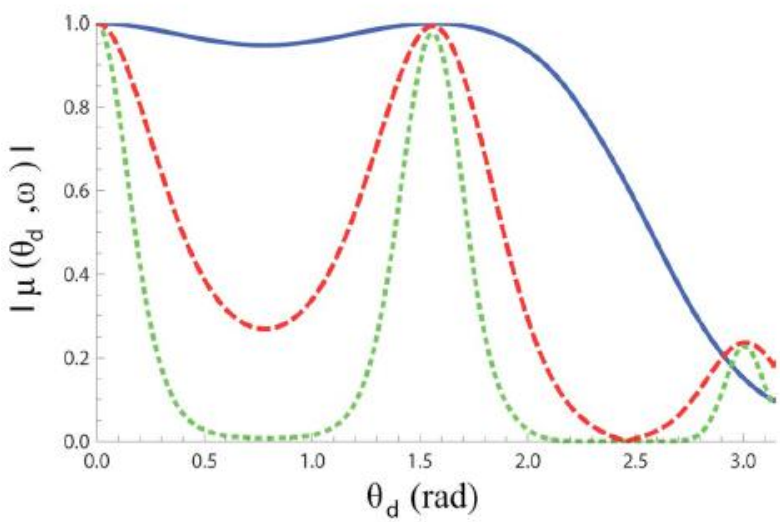


$$\eta(\mathbf{K}_2-\mathbf{K}_1) = \exp\left[-\frac{(\mathbf{K}_2-\mathbf{K}_1)^2}{(k\delta)^2}\right]$$

Effect of the correlation width δ of the pair-structure factor on the spectral density and the degree of coherence

$k\delta = 0.1$ $\qquad$ $k\delta = 1$

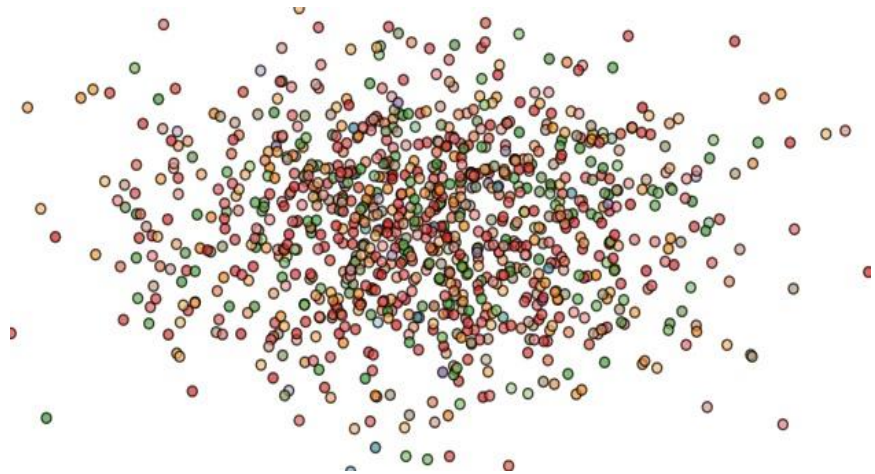




Collections of particles of different types

- Scattering potential of the medium

$$F(\mathbf{r}_1,\omega)=\begin{cases}\dfrac{1}{4\pi}k^2[n^2(\mathbf{r}_1,\omega)-1] & \mathbf{r}_1\in V\\ 0 & \mathbf{r}_1\notin V\end{cases}$$



- The correlation function of the total scattering potential

$$C_F(\mathbf{r}_1,\mathbf{r}_2,\omega)=\left\langle F^*(\mathbf{r}_1)F(\mathbf{r}_2)\right\rangle$$

- A scatterer consisting of different types of particles. The total scattering potential is

$$F(\mathbf{r}',\omega)=\sum_{l=1}^{L}\sum_{m=1}^{M_l}f_l(\mathbf{r}'-\mathbf{r}_{lm},\omega)$$

- The Fourier transform of the correlation function of the total scattering potential

$$\widetilde{C}_F(-\mathbf{K}_1,\mathbf{K}_2,\omega)=\mathbf{F}^\dagger(\mathbf{K}_1,\omega)\mathbf{M}^{(L)}(\mathbf{K}_1,\mathbf{K}_2,\omega)\mathbf{F}(\mathbf{K}_2,\omega)$$

$$\mathbf{F}(\mathbf{K},\omega)=(\widetilde{f}_1(\mathbf{K},\omega),\widetilde{f}_2(\mathbf{K},\omega),\ldots,\widetilde{f}_L(\mathbf{K},\omega))^T \qquad \widetilde{f}_l(\mathbf{K},\omega)=\int_V f_l(\mathbf{r}',\omega)\mathrm{e}^{-i\mathbf{K}\cdot\mathbf{r}'}\,\mathrm{d}^3r'$$

Pair-structure matrix

- $\mathbf{M}^{(L)}(\mathbf{K}_1, \mathbf{K}_2, \omega)$ is pair-structure matrix which contains the information about cross-correlation properties of different particle types.

$$\mathbf{M}^{(L)}(\mathbf{K}_1, \mathbf{K}_2, \omega)$$

$$= \begin{pmatrix} \left\langle Q_1^*(\mathbf{K}_1,\omega) Q_1(\mathbf{K}_2,\omega) \right\rangle & \left\langle Q_1^*(\mathbf{K}_1,\omega) Q_2(\mathbf{K}_2,\omega) \right\rangle & \dots & \left\langle Q_1^*(\mathbf{K}_1,\omega) Q_L(\mathbf{K}_2,\omega) \right\rangle \\ \left\langle Q_2^*(\mathbf{K}_1,\omega) Q_1(\mathbf{K}_2,\omega) \right\rangle & \left\langle Q_2^*(\mathbf{K}_1,\omega) Q_2(\mathbf{K}_2,\omega) \right\rangle & \dots & \left\langle Q_2^*(\mathbf{K}_1,\omega) Q_L(\mathbf{K}_2,\omega) \right\rangle \\ \dots & \dots & \dots & \dots \\ \left\langle Q_L^*(\mathbf{K}_1,\omega) Q_1(\mathbf{K}_2,\omega) \right\rangle & \left\langle Q_L^*(\mathbf{K}_1,\omega) Q_2(\mathbf{K}_2,\omega) \right\rangle & \dots & \left\langle Q_L^*(\mathbf{K}_1,\omega) Q_L(\mathbf{K}_2,\omega) \right\rangle \end{pmatrix}$$

$$Q_l(\mathbf{K}) = \sum_{m=1}^{M_l} \exp[-i\mathbf{K} \cdot \mathbf{r}_{lm}]$$

$$\mu(r\mathbf{s}_1, r\mathbf{s}_2, \omega) = \frac{\mathbf{F}^\dagger(\mathbf{K}_1,\omega)\mathbf{M}^{(L)}(\mathbf{K}_1,\mathbf{K}_2,\omega)\mathbf{F}(\mathbf{K}_2,\omega)}{\sqrt{\mathbf{F}^\dagger(\mathbf{K}_1,\omega)\mathbf{M}^{(L)}(\mathbf{K}_1,\mathbf{K}_1,\omega)\mathbf{F}(\mathbf{K}_1,\omega)}\sqrt{\mathbf{F}^+(\mathbf{K}_2,\omega)\mathbf{M}^{(L)}(\mathbf{K}_2,\mathbf{K}_2,\omega)\mathbf{F}(\mathbf{K}_2,\omega)}}$$

- It indicates the dependence of the state of coherence of scattered light on the correlation properties of the particles within one type and across different types.

- In the case when $M_{ij}^{(L)}(\boldsymbol{K},\boldsymbol{K},\omega) \approx M^{(L)}(\boldsymbol{K},\boldsymbol{K},\omega)$

$$\mu(r\boldsymbol{s}_1, r\boldsymbol{s}_2, \omega) = \frac{\sum_{i=1}^{L}\sum_{j=1}^{L} \tilde{f}_i^{*}(\boldsymbol{K}_1,\omega)\tilde{f}_j(\boldsymbol{K}_2,\omega) m_{ij}^{(L)}(|\boldsymbol{K}_1-\boldsymbol{K}_2|,\omega)}{\left|\sum_{i=1}^{L}\sum_{j=1}^{L} \tilde{f}_i(\boldsymbol{K}_1,\omega)\tilde{f}_j(\boldsymbol{K}_2,\omega)\right|}$$

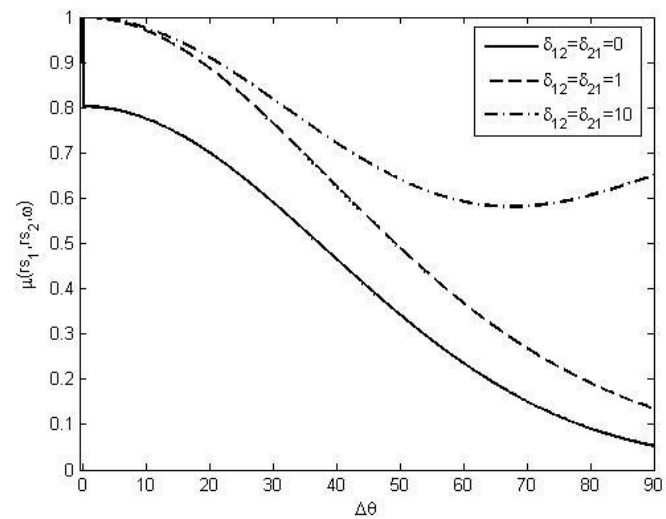
$$m_{ij}^{(L)}(|\boldsymbol{K}_1-\boldsymbol{K}_2|,\omega) = \frac{M_{ij}^{(L)}(\boldsymbol{K}_1,\boldsymbol{K}_2,\omega)}{\sqrt{M_{ij}^{(L)}(\boldsymbol{K}_1,\boldsymbol{K}_1,\omega)}\sqrt{M_{ij}^{(L)}(\boldsymbol{K}_2,\boldsymbol{K}_2,\omega)}}$$

the degree of angular correlation of the collection of particles

- Assume that

$$m_{ij}^{(2)}(|\boldsymbol{K}_1-\boldsymbol{K}_2|,\omega) = \exp\left[-\frac{|\boldsymbol{K}_1-\boldsymbol{K}_2|^2}{(k\delta_{ij})^2}\right]$$

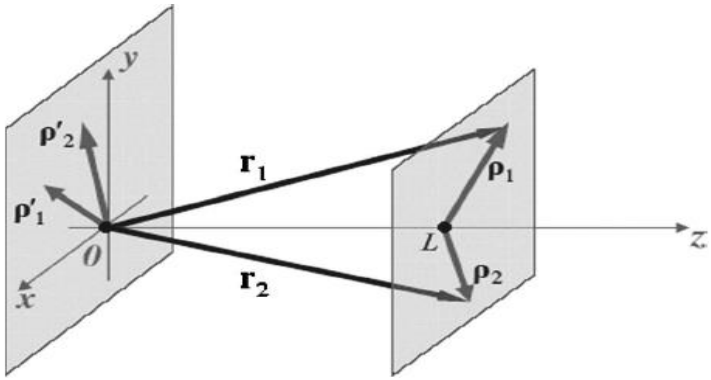
δ_{ij} the correlation strength between the collection of particles



Propagation through atmospheric turbulence

- In atmospheric turbulence, we write

$$U^{(i)}(\boldsymbol{r},\omega) = \int a^{(i)}(\boldsymbol{s},\omega) P_{\mathbf{s}}^{T}(\boldsymbol{r},\omega)\mathrm{d}^2 s_{\perp}$$



$P_{\mathbf{s}}^{T}(\boldsymbol{r},\omega)$ represents a plane wave distorted by the random medium (atmospheric turbulence)

- The cross-spectral density function

$$W^{(i)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = \int \left\langle a^{(i)*}(\boldsymbol{s}_1,\omega) a^{(i)}(\boldsymbol{s}_2,\omega)\right\rangle \left\langle P_{\mathbf{s}_1}^{T*}(\boldsymbol{r}_1,\omega) P_{\mathbf{s}_2}^{T}(\boldsymbol{r}_2,\omega)\right\rangle d^2 s_{1\perp} d^2 s_{2\perp}$$

- By a Rytov series, we approximate the distorted plane wave

$$P_{\mathbf{s}}^{T}(\boldsymbol{r},\omega) = \exp(ik\boldsymbol{s}\cdot\boldsymbol{r})\exp[\psi_{\mathbf{s}}^{(1)}(\boldsymbol{r},\omega) + \psi_{\mathbf{s}}^{(2)}(\boldsymbol{r},\omega) + \ldots]$$

$$\left\langle P_{\mathbf{s}_1}^{T*}(\boldsymbol{r}_1,\omega) P_{\mathbf{s}_2}^{T}(\boldsymbol{r}_2,\omega)\right\rangle = \exp[ik(\boldsymbol{s}_2\cdot\boldsymbol{r}_2 - s_1\cdot\boldsymbol{r}_1)]\exp\left[2E_{\mathbf{s}_1,\mathbf{s}_2}^{(1)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) + E_{\mathbf{s}_1,\mathbf{s}_2}^{(2)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)\right]$$

Propagation through atmospheric turbulence

- In the case when the atmospheric turbulence is isotropic,

$$E^{(1)}_{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1,\omega) = -2\pi^2 k^2 \int_0^L \mathrm{d}z \int_0^\infty \kappa \mathrm{d}\kappa \Phi_n(z,\kappa)$$

$$E^{(2)}_{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1,\omega) = 4\pi^2 k^2 \int_0^L \mathrm{d}z \int_0^\infty \kappa \mathrm{d}\kappa \Phi_n(z,\kappa) J_0\left[\kappa\left|(\boldsymbol{r}_{2\perp} - \boldsymbol{r}_{1\perp}) - (L-z)(\boldsymbol{s}_{2\perp} - \boldsymbol{s}_{1\perp})\right|\right]$$

- With von Karman spectrum

$$\Phi_n(\kappa) = 0.033 C_n^2 \frac{\exp\left[-\kappa^2/\kappa_m^2\right]}{(\kappa^2 + \kappa_0^2)^{11/6}}$$

- With mathematical formula

$$\int_0^L dz J_0\left[\kappa z\left|\boldsymbol{s}_2 - \boldsymbol{s}_1\right|\right] = \frac{1}{\kappa\left|\boldsymbol{s}_2 - \boldsymbol{s}_1\right|}\int_0^{\kappa L\left|\boldsymbol{s}_2 - \boldsymbol{s}_1\right|} J_0(t)dt$$

$$\int_0^z J_0(t)dt = \sum_{k=0}^{\infty} 2J_{2k+1}(z)$$

$$E^{(1)}_{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1,\omega) = -2\pi^2 \times 0.033 C_n^2 k^2 L \times 27.8495$$

$$E^{(2)}_{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{r}_2 - \boldsymbol{r}_1,\omega) = 4\pi^2 \times 0.033 C_n^2 k^2 \times \frac{2}{\left|\boldsymbol{s}_{2\perp} - \boldsymbol{s}_{1\perp}\right|}\sum_{m=0}^{\infty}\left[\int_0^\infty \mathrm{d}\kappa \frac{\exp\left[-\kappa^2/\kappa_m^2\right]}{(\kappa^2 + \kappa_0^2)^{11/6}} J_{2m+1}\left[\kappa L\left|\boldsymbol{s}_{2\perp} - \boldsymbol{s}_{1\perp}\right|\right]\right]$$

Propagation through atmospheric turbulence

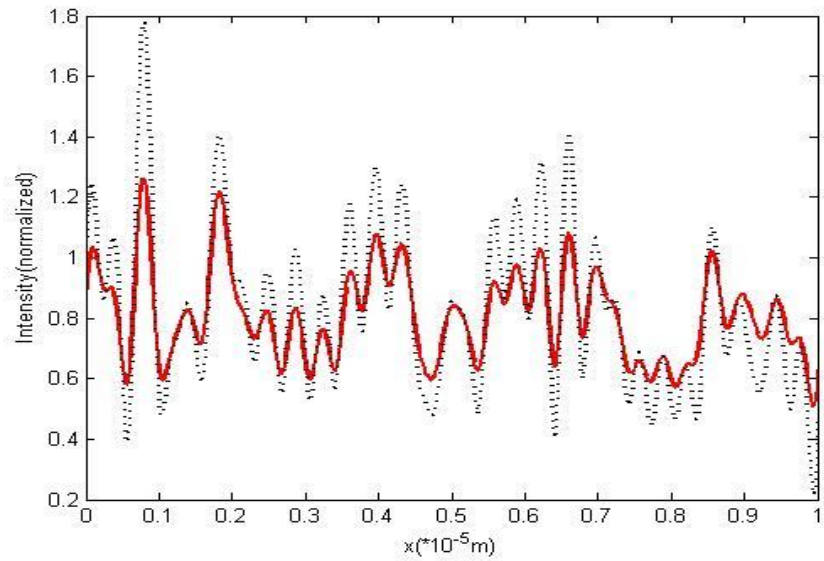
- Considering the impact of the scatterer and the turbulence

$$U^{(s)}(\boldsymbol{r},\omega)=\int a^{(s)}(\boldsymbol{s},\omega)P_{\mathbf{s}}^{T}(\boldsymbol{r},\omega)d^{2}s_{\perp}$$

cross-spectral density function of the field after transmission through turbulent and particulate random medium

$$W^{(s)}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)=\left(\frac{k}{2\pi}\right)^{2}\iint\frac{1}{s_{1z}s_{2z}}\iint A^{(i)}(\boldsymbol{s}_1',\boldsymbol{s}_2',\omega)C_F\left[-\boldsymbol{K}_1,\boldsymbol{K}_2,\omega\right]d^{2}s'_{1\perp}d^{2}s'_{2\perp}$$

$$\times\exp[ik(\boldsymbol{s}_2\cdot\boldsymbol{r}_2-\boldsymbol{s}_1\cdot\boldsymbol{r}_1)]\exp\left[2E^{(1)}_{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)+E^{(2)}_{\mathbf{s}_1,\mathbf{s}_2}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)\right]d^{2}s_{1\perp}d^{2}s_{2\perp}$$

- The von Karman spectrum

$$\Phi_n(\kappa)=0.033C_n^2\frac{\exp\left[-\kappa^2/\kappa_m^2\right]}{(\kappa^2+\kappa_0^2)^{11/6}}$$