MGSM beam model

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sqrt{S(\boldsymbol{\rho}_1)}\sqrt{S(\boldsymbol{\rho}_2)}\mu(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) \quad (1)$$

$$S(\boldsymbol{\rho}) = \exp\left[-\frac{\rho^2}{2\sigma^2}\right]$$

$$\mu(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2) = \frac{1}{C_0}\sum_{m=1}^{M}\binom{M}{m}\frac{(-1)^{m-1}}{m}\exp\left[-\frac{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^2}{2m\delta^2}\right]$$

O. Korotkova, S. Sahin, and E. Shchepakina, “Multi-Gaussian Schell-model beams,” J. Opt. Soc. Am. A **29**, 2159-2164 (2012).

Discrete MGSM beam model (1)

Note that

$$\exp\left[-\frac{|\boldsymbol{\rho}_1-\boldsymbol{\rho}_2|^2}{2m\delta^2}\right]=\int_{-\infty}^{\infty}\pi 2m\delta^2\exp(-\pi^2 2m\delta^2 f^2)\exp[j2\pi\boldsymbol{f}\cdot(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)]d^2f \quad (2)$$

On substituting Eq. 2 into Eq. 1 and exchanging summation over m and integral over $\boldsymbol{f}$

$$W(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)=\sqrt{S(\boldsymbol{\rho}_1)}\sqrt{S(\boldsymbol{\rho}_2)}\mu(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2) \quad (3)$$

$$=\int_{-\infty}^{\infty}A(\boldsymbol{\rho}_1,\boldsymbol{f})A^*(\boldsymbol{\rho}_2,\boldsymbol{f})d^2f$$

where

$$A(\boldsymbol{\rho},\boldsymbol{f})=\sqrt{\frac{1}{C_0}\sum_{m}^{M}\binom{M}{m}(-1)^{m-1}2\pi\delta^2\exp(-\pi^2 2m\delta^2 f^2)\exp\left(-\frac{\rho^2}{\sigma^2}\right)}\exp(j2\pi\boldsymbol{f}\cdot\boldsymbol{\rho}) \quad (4)$$

Eqs. (3) and (4) suggest that a MGSM beam can be treated as incoherent superposition of beamlet $A(\boldsymbol{\rho},\boldsymbol{f})$, a Gaussian beam whose amplitude is a function of its direction characterized by $\boldsymbol{f}$.

Discrete MGSM beam model (2)

The scintillation properties of a partially coherent beam can be characterized by its discrete beamlet array *

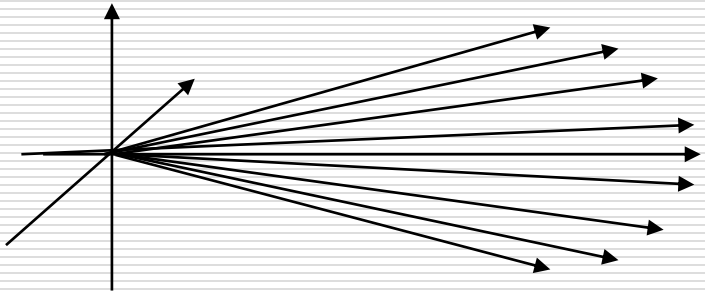
$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int_{-\infty}^{\infty} A(\boldsymbol{\rho}_1, \boldsymbol{f}) A^*(\boldsymbol{\rho}_2, \boldsymbol{f}) d^2 f$$

$$\approx \sum_{|\boldsymbol{u}_\perp|<1} A(\boldsymbol{\rho}_1, \boldsymbol{u}_\perp) A^*(\boldsymbol{\rho}_2, \boldsymbol{u}_\perp)$$

$$\boldsymbol{f} = \frac{k\boldsymbol{u}_\perp}{2\pi} \quad k = \frac{2\pi}{\lambda}$$

$$\boldsymbol{u}_\perp = (u_{\perp x}, u_{\perp y})$$

$\boldsymbol{u}_\perp$ is the projection of the unit vector $\boldsymbol{u}$ on the source plane.
$\boldsymbol{u}$ specifies the direction of $A(\boldsymbol{\rho}_1, \boldsymbol{u}_\perp)$



Source plane

Discrete MGSM beam
The amplitudes of its constituent beamlets form a flat-top function in the $\boldsymbol{u}_\perp$ plane.

Because evanescent modes do not propagate, only $|\boldsymbol{u}_\perp| < 1$ are included in the discrete MGSM

* Y. Gu and G. Gbur, “Scintillation of pseudo-Bessle correlated beams in atmospheric turbulence,” J. Opt. Soc. Am. A **27**, 2621-2629 (2010).

Scintillation of discrete MGSM beams: analytical model in weak turbulence

The Rytov approximation

$$U(\vec{\rho},L)=U_0(\vec{\rho},L)\exp[\psi(\vec{\rho},L)]$$

$\psi(\vec{\rho},L)$ is a turbulence induced complex phase perturbation.

$$\psi(\vec{\rho},L)=\psi_1(\vec{\rho},L)+\psi_2(\vec{\rho},L)+\cdots$$

The scintillation index of a discrete MGSM beam:

$$c^2(\vec{\rho},L)=\frac{<I^2>}{<I>^2}-1$$

$$=\frac{\sum_{n=1}^{N}\sum_{m=1}^{N}M_{nm}(\vec{\rho},L)}{\left[\sum_{n=1}^{N}<I_n(\vec{\rho},L)>\right]^2}$$

where

$$M_{nm}(\vec{\rho},L)=<I_n(\vec{\rho},L)><I_m(\vec{\rho},L)>\sigma_{nm}^2(\vec{\rho},L)$$

$$\sigma_{nm}^2=\frac{<I_n(\vec{\rho},L)I_m(\vec{\rho},L)>}{<I_n(\vec{\rho},L)><I_m(\vec{\rho},L)>}-1$$

$$<I_n(\vec{\rho},L)>=I_{0n}(\vec{\rho},L)\exp\left[E_2^{nn}(\vec{\rho},L)+2\,\mathrm{Re}\,E_1^{nn}(\vec{\rho},L)\right]$$

$$<I_n(\vec{\rho},L)I_m(\vec{\rho},L)>=<I_n(\vec{\rho},L)><I_m(\vec{\rho},L)>$$
$$\times\exp\left[2\,\mathrm{Re}\,E_2^{nn}(\vec{\rho},L)+2\,\mathrm{Re}\,E_3^{nn}(\vec{\rho},L)\right]$$

Scintillation of discrete MGSM beams: analytical model in weak turbulence

$$E_1^{nn}(\vec{\rho},L) = <\psi_{n2}(\vec{\rho},L)> + \frac{1}{2}<\psi_{n1}^2(\vec{\rho},L)>$$

$$= -\pi k^2 \exp\left[\frac{ikLu_{\perp n}^2}{2p(L)}\right]\int_0^L dz \exp\left[\frac{ikzu_{\perp n}^2}{2p(z)}\right]\exp\left[-\frac{ik\gamma(L-z)u_{\perp n}^2}{2p^2(z)}\right]\int d^2\kappa\Phi_n(\kappa)$$

$$E_2^{nm}(\vec{\rho},L) = <\psi_{n1}(\vec{\rho},L)\psi_{m1}^*(\vec{\rho},L)>$$

$$= 2\pi k^2 \exp\left[\frac{ikLu_{\perp n}^2}{2p(L)}\right]\exp\left[-\frac{ikLu_{\perp m}^2}{2p^*(L)}\right]$$

$$\times\int_0^L dz \exp\left[-\frac{ikzu_{\perp n}^2}{2p(z)} + \frac{ikzu_{\perp m}^2}{2p^*(z)}\right]\exp\left[-\frac{ik\gamma(L-z)u_{\perp n}^2}{2p^2(z)} - \frac{ik\gamma^*(L-z)u_{\perp m}^2}{2(p^2(z))^*}\right]$$

$$\times\int d^2\kappa\Phi_n(\kappa)\exp\left[-\frac{i(\gamma-\gamma^*)(L-z)\kappa^2}{2k}\right]\exp\left[i\vec{\kappa}\cdot(\gamma\vec{\rho}-\gamma^*\vec{\rho})\right]\exp\left[-i(L-z)\vec{\kappa}\cdot\left(\frac{\vec{u}_{\perp n}}{p(L)} - \frac{\vec{u}_{\perp m}}{p^*(L)}\right)\right]$$

Scintillation of discrete MGSM beams: analytical model in weak turbulence

$$E_3^{nm}(\vec{\rho},L) = <\psi_{n1}(\vec{\rho},L)\psi_{m1}(\vec{\rho},L)>$$

$$= -2\pi k^2 \exp\left[\frac{ikLu_{\perp n}^2}{2p(L)}\right]\exp\left[\frac{ikLu_{\perp m}^2}{2p(L)}\right]$$

$$\times \int_0^L dz \exp\left[-\frac{ikz}{2p(z)}\left(u_{\perp n}^2 + u_{\perp m}^2\right)\right]\exp\left[-\frac{ik\gamma(L-z)}{2p^2(z)}\left(u_{\perp n}^2 + u_{\perp m}^2\right)\right]$$

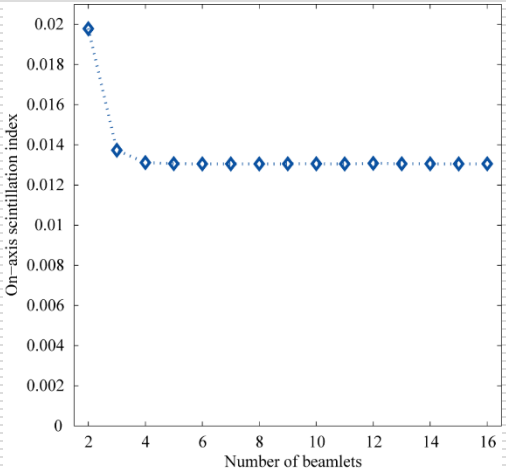
$$\times \int d^2\kappa \Phi_n(\kappa)\exp\left[-\frac{i\gamma(L-z)\kappa^2}{k}\right]\exp\left[-i\frac{(L-z)}{p(L)}\vec{\kappa}\cdot(\vec{u}_{\perp n} - \vec{u}_{\perp m})\right]$$

where $p(L) = 1 + i\alpha L,\ \alpha = \dfrac{2}{k\sigma^2},\ \gamma = \dfrac{p(z)}{p(L)}$

$\Phi_n(\kappa)$ is the turbulence spectrum function.

Please see Gu and Gbur JOSA A 2010 for the detailed derivation of equations in page 4-6.

Scintillation of discrete MGSM beams: how many beamlets are sufficient

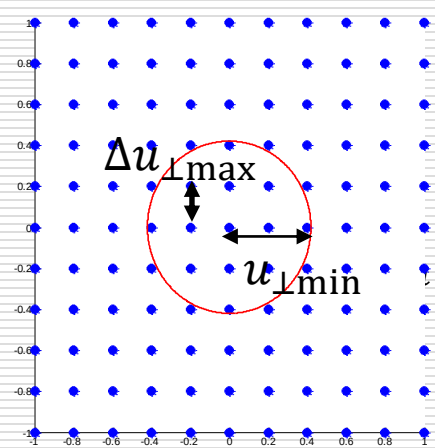


The scintillation properties of a Bessel correlated beam can be characterized by its discrete beamlet array with as less as 4 beamlets*

* Fig. 3 in Y. Gu and G. Gbur, "Scintillation of pseudo-Bessle correlated beams in atmospheric turbulence," J. Opt. Soc. Am. A **27**, 2621-2629 (2010).

how many beamlets are sufficient for a MGSM beam?

- Determine the minimum sampling radius $u_{\perp\min}$ in the $\boldsymbol{u}_\perp$ plane
- Determine the maximum sampling separation $\Delta u_{\perp\max}$ in the $\boldsymbol{u}_\perp$ plane
- Obtain the number of beamlets bounded within the circle in the $\boldsymbol{u}_\perp$ plane



Scintillation of discrete MGSM beams: general steps

- Usually, we should start with a large sampling radius and a small sampling separation so that the discrete MGSM beam is similar to the corresponding continus MGSM beam.

- Sampled $\boldsymbol{u}_\perp = (u_{\perp x}, u_{\perp y})$ distribute in the 2D $\boldsymbol{u}_\perp$ plane, resulting a large number of beamlets, then heavy computation load.

- Since both the turbulence and the MGSM beam are isotropic, we only sample $u_{\perp x}$ but let $u_{\perp y} = 0$. So the direction vectors of the beamlets are in xoz plane. This the is so called 1D discrete MGSM beam.

- We then determine $u_{\perp min}$ and $\Delta u_{\perp max}$ of the 1D discrete MGSM beam.

Scintillation of discrete MGSM beams: parameters

$\lambda = 0.633\ \mu\mathrm{m}$

$L = 323\ \mathrm{m}$

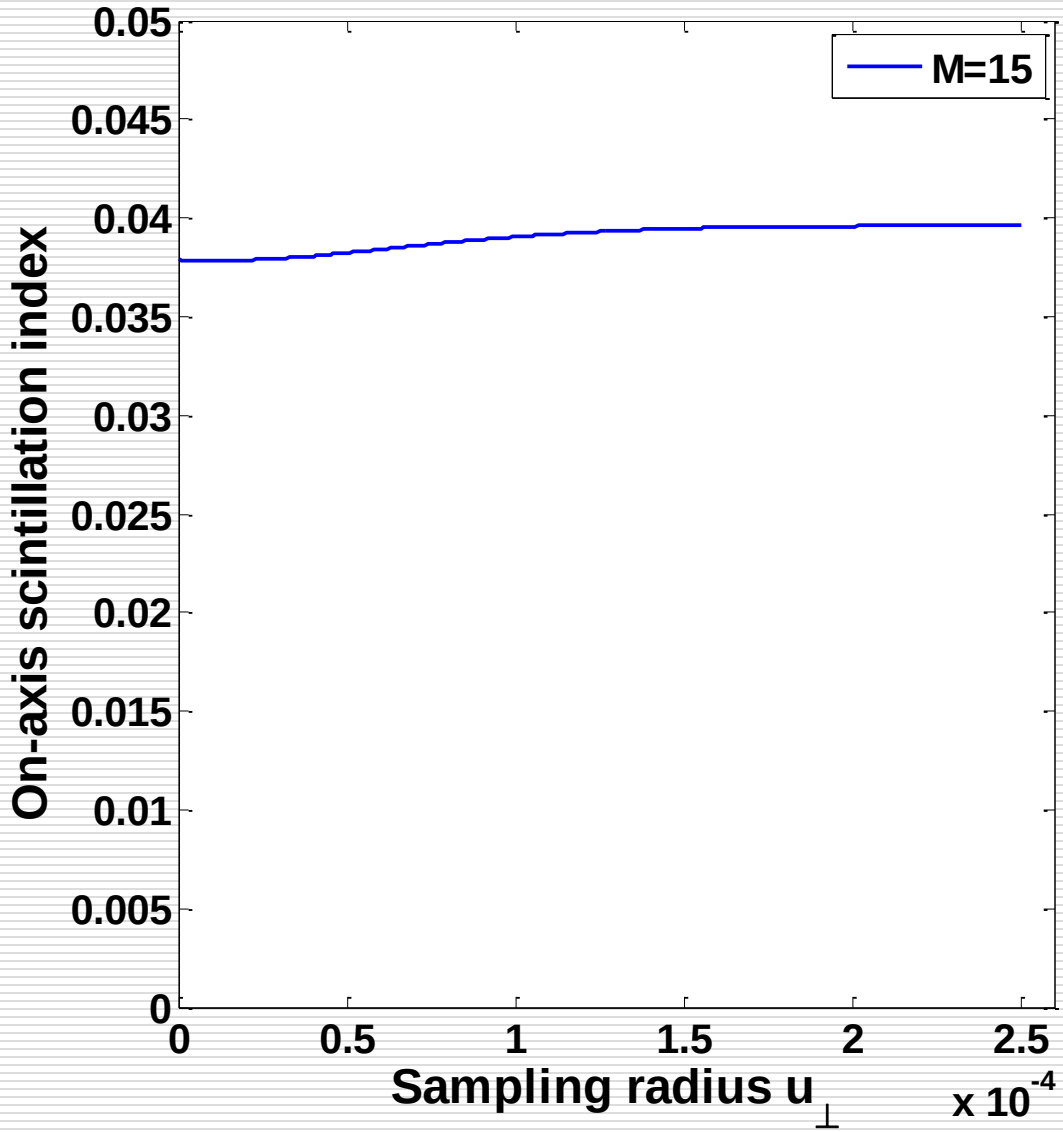
$C_n^2 = 10^{-14}\ \mathrm{m}^{-2/3}$ the inner scale is 1 mm and the outer scale is 10 m

$\delta = 0.27\ \mathrm{mm}$ (according to Eq. 9 in the proceeding for SPIE Optics & Photonics 2014)

$\sigma = 1.8\ \mathrm{mm}$ the mechanical iris to truncate the beam was around $0.5\ \mathrm{cm}$ in diameter

Then the width σ of the soft-Gaussian $\exp\left(-\frac{\rho^2}{\sigma^2}\right)$ is $0.5/\left(2\sqrt{2}\right)$ After O. Korotkova, "Changes in the intensity fluctuations of a class of random electromagnetic beams on propagation," J. Opt. A: Pure Appl. Opt., 8, 30-37 (2006) (in particular the lines after Eq. 29 on page 34.)

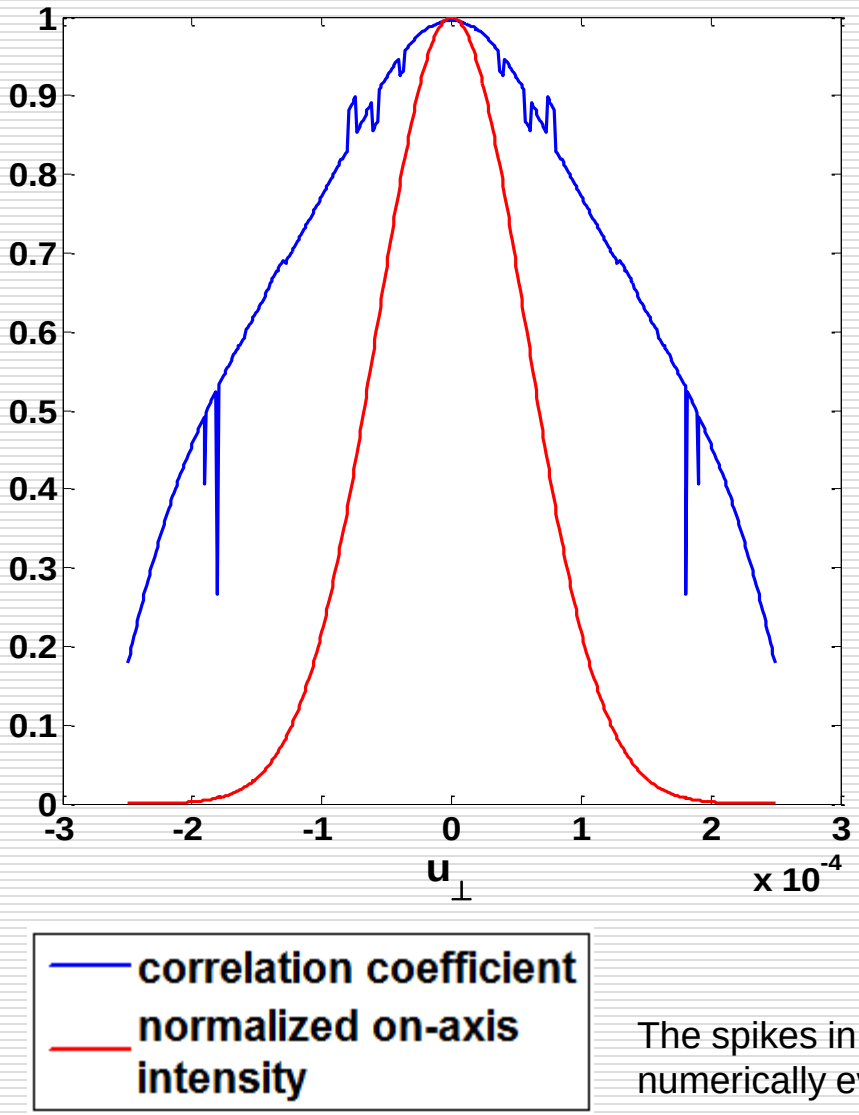
Scintillation of discrete MGSM beams: 1D



$\Delta u_{\perp x} = 10^{-6}$, so 501 beamlets are included when the sampling radius is 2.5×10^{-6}. When the sampling radius is 0, it corresponds to a fully coherent Gaussian beam, since only one mode is included [$\boldsymbol{u}_{\perp} = (0,0)$].

From the left figure, it looks like that the on-axis scintillation index is almost invariant when we add more incoherent beamlets. In other word, a 1D partially coherent MSGM beam has s

Explanation of Figure on page 10



Note that the width of the Gaussian beamlets on the source plane is 1.8mm. After propagating for 323 m, the corresponding width is 3.7cm, 20 times wider than the size on the source plane. Under this circumstance, the beamlets are highly correlated.

The blue curve in the left figure is the correlation coefficient between the on-axis intensity of the beamlet $[\boldsymbol{u}_\perp = (0,0)]$ and the on-axis intensity of the other beamlets. It can be seen $\boldsymbol{u}_\perp = (0,0)$ is highly correlated with its neighbors. On the other side, the on-axis intensity approach zero for beamlets with large $|\boldsymbol{u}_\perp|$. Their contribution is negligible though they are low correlated with $\boldsymbol{u}_\perp = (0,0)$. So the on-axis scintillation index plot shown ong page 10 is almost invariant when we include more beamlets.

The spikes in the blue curve is probably because of the errors of numerically evaluation of the equations on pages 4-6 (using Mathematica).

Scintillation of discrete MGSM beams: 1D used in SPIE

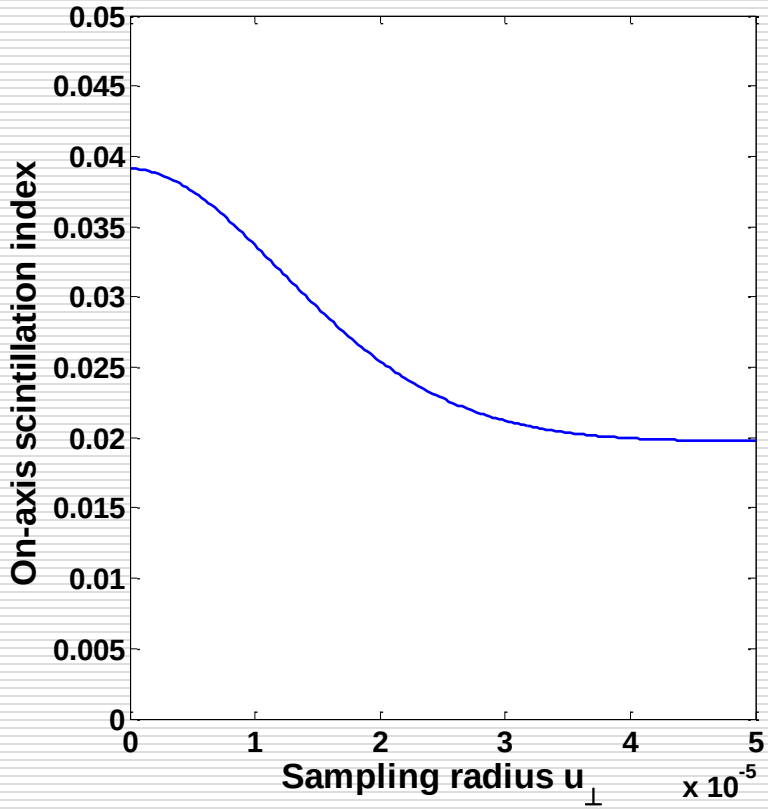
$\lambda = 1.55\ \mu\text{m}$

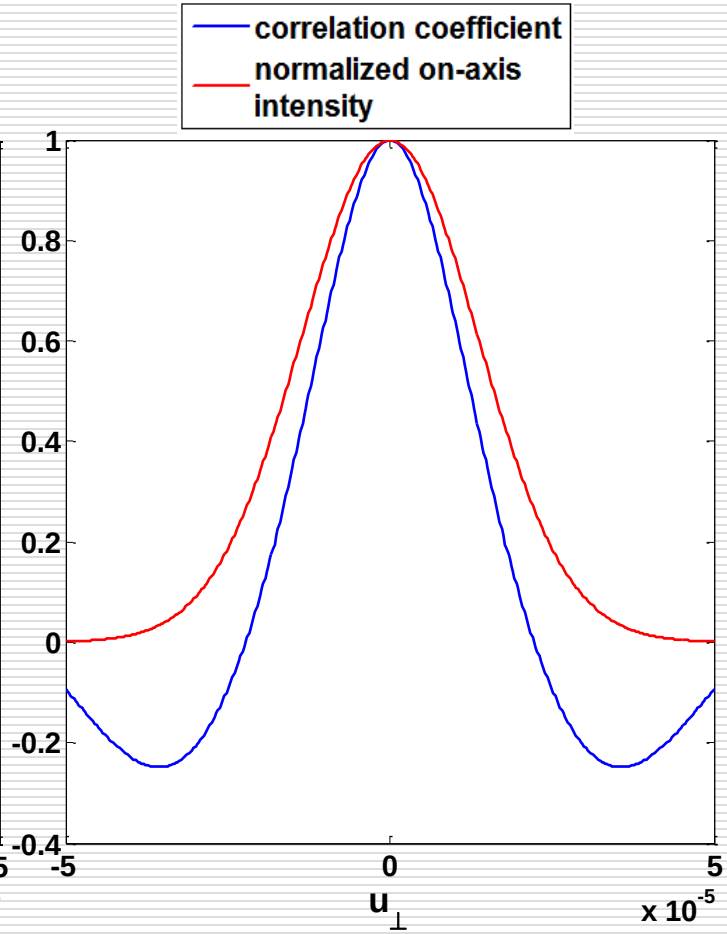
$L = 2\ \text{km}$

$\sigma = 5\ \text{cm}$

$\delta = 1.08\ \text{cm}$

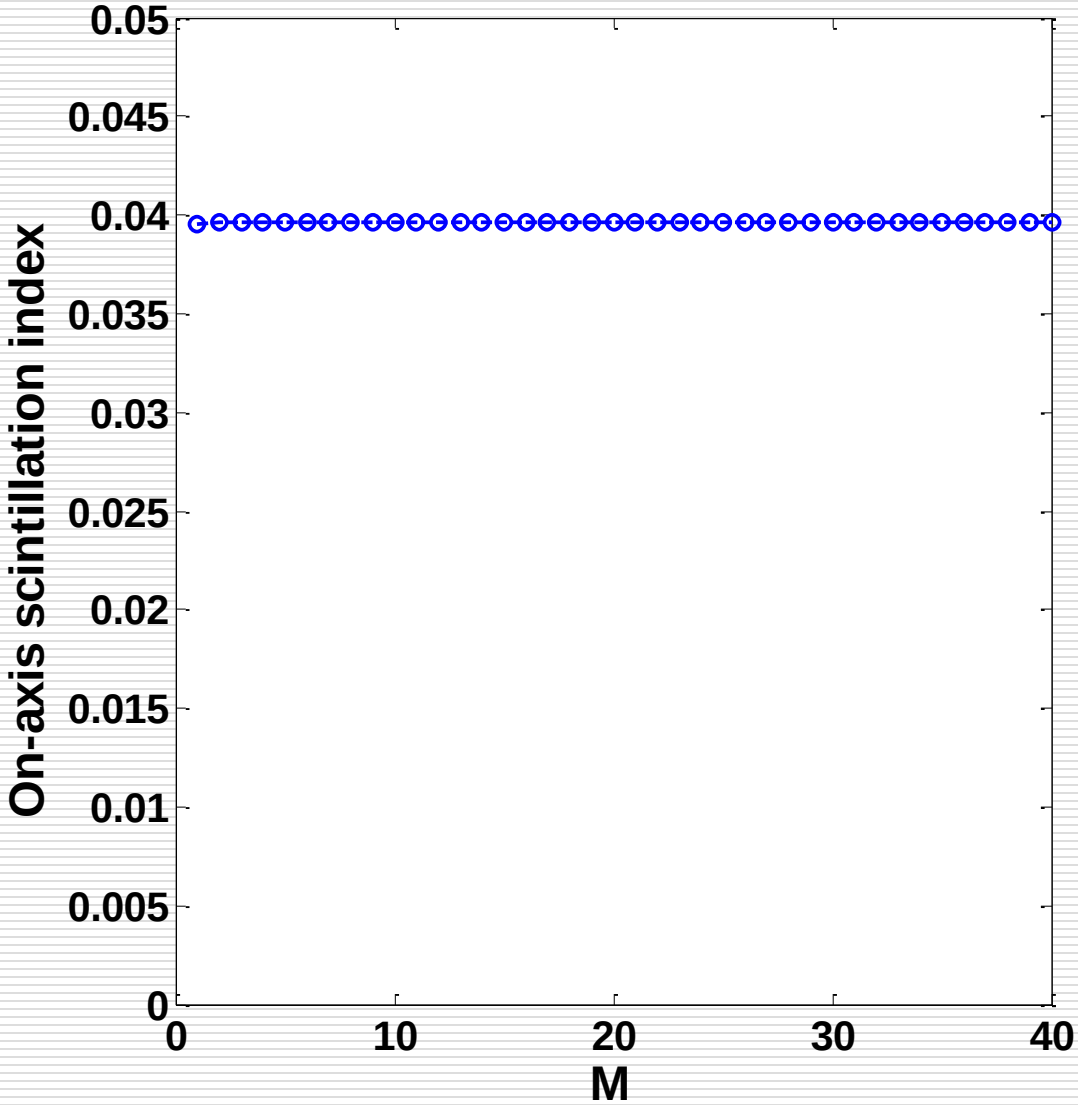
$C_n^2 = 10^{-15}\ \text{m}^{-2/3}$





The beam example used in the SPIE paper has a wide width. So after propagating for 2km, the width is 5.38 cm. The correlation curve decreases relatively quick. Even negative correlation appears in the right figure. In addition, the correlation curve has a comparable width with the normalized on-axis intensity curve

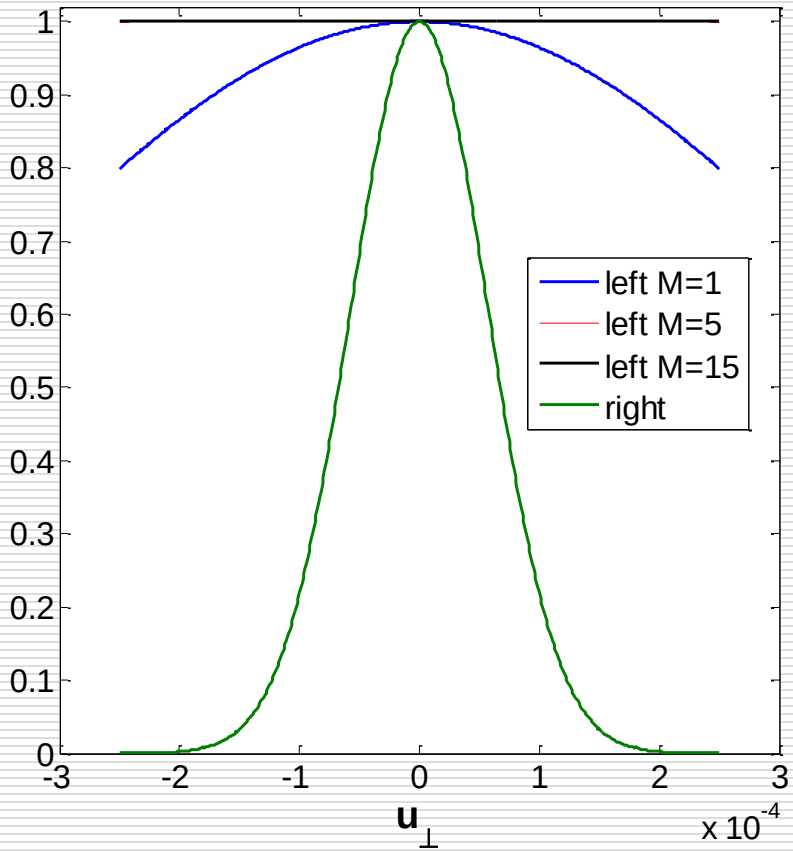
Scintillation of discrete MGSM beams: 1D



From the left figure, it looks like that the on-axis scintillation index is almost invariant when we change M.

Explanation of Figure on page 13

$$A(\boldsymbol{\rho},\boldsymbol{f}) = \left[\sqrt{\frac{1}{C_0}\sum_{m}^{M}\binom{M}{m}(-1)^{m-1}2\pi\delta^2\exp(-\pi^2 2m\delta^2 f^2)}\right]\left[\exp\left(-\frac{\rho^2}{\sigma^2}\right)\exp(j2\pi\boldsymbol{f}\cdot\boldsymbol{\rho})\right] \quad (4)$$



From Eq. 4 (beamlet on the source plane), the on-axis intensities of these beamlets on the detector plane are determined by two parts (left: a flat-top function and right: a Gaussian function).

The normalized left part (different M) and the right part for the parameters on page 9 are shown in the left figure.

For such a low coherent MGSM beam, the on-axis intensities of the bealets are determined by the right part, independent of the left part

Scintillation of discrete MGSM beams: 1D used in SPIE

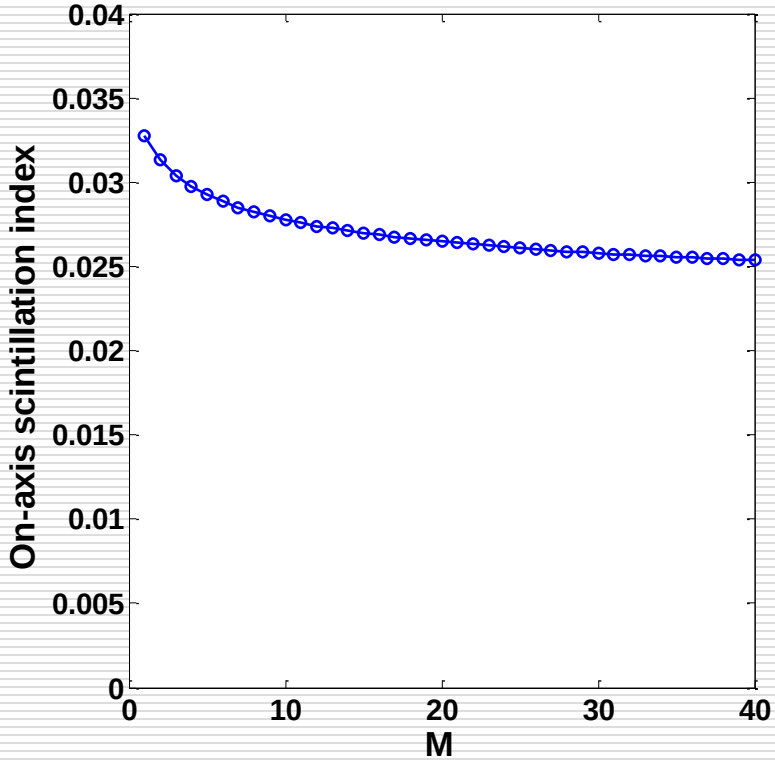
$\lambda = 1.55\ \mu\text{m}$

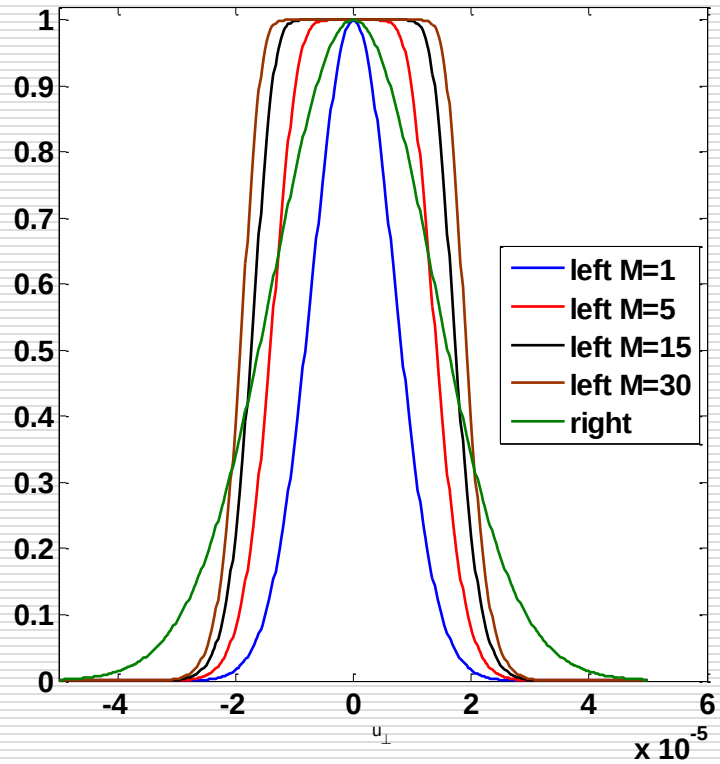
$L = 2\ \text{km}$

$\sigma = 5\ \text{cm}$

$\delta = 3.54\ \text{cm}$

$C_n^2 = 10^{-15}\ \text{m}^{-2/3}$





The beam example used in the SPIE paper has a large coherence length. So after propagating for 2km, the normalized left part (different M) has comparable width with the normalized right part. So the on-axis scintillation index of 1D MGSM beam varies as a function of M.