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Integration: Basic Definitions, Techniques and Properties

An integral is a sum of a large number of small contributions. The critical consideration is that, in the limit that the contributions become smaller and more numerous, the sum converges to a defined value.

\[
\sum f_i \Delta x_i < \text{Area} = \int_a^b f(x) \, dx < \sum f_i^{\text{max}} \Delta x_i
\]

Figure IB.1: The Riemann Integral

The figure above depicts two sums that approximate the area under the \( f(x) \) curve between \( a \) and \( b \). The interval between \( a \) and \( b \) is divided into \( N \) equal width sub-intervals. An upper sum is computed by taking the largest value of the function in each interval, multiplying it by the width of the bin and summing. A lower sum is computed in an analogous manner using the smallest value of the function in each interval. In the limit that \( N \) increases indefinitely, these procedures yields sequences of upper and lower sums. If both sequences converge and they converge to the same value, then the function is Riemann integrable from \( a \) to \( b \). The Riemann integral of a finite function with a finite number of discontinuities over a finite range exists. There are alternative definitions of integration that are less restrictive. The Riemann integral is, however, sufficient for our immediate needs.

If both limits exist and are equal,
\[ \text{Limit}_{N \to \infty} \left[ \text{Lower}_N \right] = \int_a^b f(x) \, dx = \text{Limit}_{N \to \infty} \left[ \text{Upper}_N \right]. \]  

**[IB.1]**

**The Fundamental Theorem of Integral Calculus:**

If \( F(x) \) is an anti-derivative of \( f(x) \), then

\[ \int_a^b f(x) \, dx = F(b) - F(a) \]  

**[IB.2]**

This statement is equivalent to saying that integration of a function \( f(x) \) constructs an anti-derivative \( A(x) \) of that function.

\[ A(x) = \int_{x_0}^x f(x') \, dx' \]

Note that \( x' \) is a dummy-variable label, the integration label should not be also used as a limit label.

This rule is often violated in these handouts due to carelessness. Note that

\[ A(x) = \int_{x_0}^x f(x') \, dx' = \int_{x_0}^x f(s) \, ds = \int_{x_0}^x f(\zeta) \, d\zeta. \]

That is you are free to change the dummy integration label to avoid conflicts.

This theorem is to be used in the form: \( f(x) = f(x_0) + \int_{x_0}^x \left( \frac{df}{dx} \right) \, dx' \).

The fundamental theorem of integral calculus leads to precursors of Leibniz rule.

\[ A(x) = \int_a^t f(t) \, dt \quad \Rightarrow \quad \frac{dA}{dx} = f(x) \]

Adding the chain rule, \( A(x) = \int_a^{u(x)} f(t) \, dt \quad \Rightarrow \quad \frac{dA}{dx} = f(u(x)) \frac{du}{dx} \)

**Mean Value Theorem:** If a function \( f(x) \) is continuous in the interval \([a, b]\) then there exists some argument value \( c \) in that interval such that:

\[ \int_a^b f(x) \, dx = f(c)[b - a] \quad \Rightarrow \quad f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx = \left\{ f(x) \right\}_{x \in [a, b]} \]

The value \( f(c) \) is the mean value of \( f(x) \) in the interval \([a, b]\).

**Leibniz Rule:** The total derivative of an integral entails taking the derivative of the integrand as well as allowing the derivative to act on the limits of integration.

\[ \frac{d}{dt} \left[ \int_a^b f(x, t) \, dx \right] = \int_a^b \left[ \frac{\partial f}{\partial t} \right] \, dx + f(b, t) \frac{\partial b}{\partial t} - f(a, t) \frac{\partial a}{\partial t} \]

**[IB.3]**

The last two terms are surface or boundary terms and arise whenever the range of integration varies (perhaps it depends on time). A proof/motivation of Leibniz rule is presented just before the problem section of this handout.
An important 3D vector calculus application of Leibniz’s rule arises in the discussion of Faraday’s Law.

\[ \frac{d}{dt} \int_S \vec{B}(\vec{r}, t) \cdot \hat{n} \, da = \oint_C \vec{B}(\vec{r}, t) \cdot \left( \frac{d\vec{r}_C}{dt} \times d\vec{\ell} \right) + \int_S \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \cdot \hat{n} \, da \]

Note that a term \( \int_S \vec{\nabla} \cdot \vec{B} \ (\vec{\nabla} \cdot \hat{n}) \, da \) appears in the full Leibniz rule for this case, but it is discarded in the case of the magnetic flux. If you are a physics major, you should be able to justify this omission before you finish your second E&M course. See the Vector Calculus Appendix for detailed definitions of v and rC and a derivation of the rule.

The time rate of change of the magnetic flux through a thin loop of conductor is due to the addition of area elements \( \frac{d\vec{r}_C}{dt} \times d\vec{\ell} \) as the points on the conductor (or surface) move at their local velocities \( \frac{d\vec{r}_C}{dt} \) (or \( \vec{v} \)) plus the change in flux through the pre-existing area due to the magnetic field in that area changing in time. That is: the flux changes includes a motional contribution when the conductor moves in a region in which the field is non-zero, and it changes due to induction if the magnetic field itself is time-dependent.

See Wikipedia for the full version.

\[ \frac{d}{dt} \int_S \vec{F}(\vec{r}, t) \cdot \hat{n} \, da = \oint_C \frac{d\vec{r}_C}{dt} \cdot \hat{n} \, da + \int_S (\vec{\nabla} \cdot \vec{F}) \vec{v} \cdot \hat{n} \, da - \oint_{\partial \Sigma} (\vec{\nabla} \times \vec{F}) \cdot d\vec{\ell} \]

\[ \frac{d}{dt} \int_{\Sigma(t)} \vec{F}(\vec{r}, t) \cdot d\vec{A} = \int_{\Sigma(t)} (\vec{F}_t(\vec{r}, t) + [\vec{\nabla} \cdot \vec{F}(\vec{r}, t)] \vec{v} \cdot d\vec{A} - \oint_{\partial \Sigma(t)} \vec{v} \times \vec{F}(\vec{r}, t) \cdot ds \]

where:

- \( \vec{F}(\vec{r}, t) \) is a vector field at the spatial position \( \vec{r} \) at time \( t \)
- \( \Sigma \) is a moving surface in three-space bounded by the closed curve \( \partial \Sigma \)
- \( d\vec{A} \) is a vector element of the surface \( \Sigma \)
- \( ds \) is a vector element of the curve \( \partial \Sigma \)
- \( \vec{v} \) is the velocity of movement of the region \( \Sigma \)
- \( \vec{\nabla} \cdot \) is the vector divergence
- \( \times \) is the vector cross product

The double integrals are surface integrals over the surface \( \Sigma \), and the line integral is over the bounding curve \( \partial \Sigma \).
Linear Operation: Integration is a linear operation.

\[
\int [a f(x) + b g(x)] \, dx = a \int f(x) \, dx + b \int g(x) \, dx
\]

Isaac Newton (1642-1727) formulated the classical theories of mechanics and optics and invented calculus years before Leibniz. However, he did not publish his work on calculus until after Leibniz had published his version. This led to a bitter priority dispute between English and continental mathematicians which persisted for decades, to the detriment of all concerned. Newton discovered that the binomial theorem was valid for fractional powers, but left it for Wallis to publish (which he did, with credit to Newton).

Isaac Newton delayed publication of his theories of gravity until he could develop integral calculus to demonstrate that, if his proposed law for gravitation held for point masses, then, for masses outside the sphere, any spherical mass should interact just as a point mass of the total mass concentrated at the center. The delay is rumored to have been about eleven years.

Wilhelm Gottfried Leibniz (1646-1716) The last years of his life - from 1709 to 1716 - were embittered by the long controversy with John Keill, Newton, and others, as to whether he had discovered the differential calculus independently of Newton's previous investigations, or whether he had derived the fundamental idea from Newton, and merely invented another notation for it. The controversy occupies a place in the scientific history of the early years of the eighteenth century quite disproportionate to its true importance, but it so materially affected the history of mathematics in western Europe.

“The Riemann integral is the integral normally encountered in calculus texts and used by physicists and engineers. Other types of integrals exist (e.g., the Lebesque integral), but are unlikely to be encountered outside the confines of advanced mathematics texts. In fact, according to Jeffreys and Jeffreys (1988, p. 29), ”it appears that cases where these methods [i.e., generalizations of the Riemann
integral] are applicable and Riemann's [definition of the integral] is not too rare in physics to repay the extra difficulty.""


Bernhardt Riemann (1826-1866) German mathematician who studied mathematics under Gauss and physics under Wilhelm Weber. Riemann did important work in geometry, complex analysis, and mathematical physics. In his thesis, Riemann urged a global view of geometry as a study of manifolds of any number of dimensions in any kind of space. He defined space by a metric. Riemann's work laid the foundations on which general relativity was built. He also refined the definition of the integral.

© Eric W. Weisstein scienceworld.wolfram.com/biography/Riemann.html

A Guide for Integrations

In physics applications, half the battle is setting up the integral. As a rule, vector operations should be executed prior to the actual integration. Once the problem has been reduced to a specific mathematical integral, completion of the problem is usually a walkover. It is for this reason that some suggestions for attacking the setup process are presented.

1. Choose a coordinate system that is appropriate for the problem. Use symmetry as a guide.

2. Express all vectors in terms of the coordinate directions for that coordinate system, and compute all inner (dot) and cross products.

3. If a unit vector \( \hat{e} \) in the integrand is not constant with respect to the integration variables, replace that vector by its representation in terms of the constant directions \( \hat{i}, \hat{j}, \text{ and } \hat{k} \). This representation makes the dependence of the direction \( \hat{e} \) on the integration variables explicit.

By this point, the integration has been reduced to either a scalar (perhaps multiple) integral or to a set of scalar integrals multiplying constant directions. If they are present, the constant directions should be taken outside the integral leaving a sum of terms each with an integral multiplying a
constant direction. This form is a generalization of the component-wise addition of vectors. Multiple integrals are to be computed as a nested set of single integrations.

Integration Techniques

Change of variable: A first guess is to choose the argument of the most complicated function in the integrand as the new variable. Below the integration variable is changed from $t$ to $x$. The new limits are $x_{\text{lower}} = x(t_{\text{lower}})$ and $x_{\text{upper}} = x(t_{\text{upper}})$. That is the limiting values of $x$ are the values that $x$ assumes at each of the limiting values of the original integration variable. Be sure to change your limits appropriately and simultaneously!

\[ \int_{t_0}^{t'} f(x) \frac{dx}{dt'} dt' = \int_{x(t_0)}^{x(t')} f(x) \, dx \]

(Note that the dummy $t'$ is used as the integration variable as the integration variable must be distinct from the limits. The dummy variable $x$ can be used as it does not appear unmodified as a limit of integration.)

Exercise: Dummy Variable

Compute the following in a naïve fashion and compare the results. Assume $a$ is constant.

\[ \int_0^t dt \int_0^t a \, dt \quad \text{and} \quad \int_0^t dt' \int_0^t a \, dt' \quad \text{and} \quad \int_0^{t'} dt' \int_0^{t'} a \, dt' \]

Based on your experiences in introductory mechanics, which form represents the desired operation? Discuss the necessity to use dummy variable and to properly nest integrals when appropriate.

Add conventions nesting and display constant acceleration result

More detail – change of variable: Adopting new notation, the goal is to change from an integral with respect to $x$ to an integral with respect to $u$ that has the same value and that works independent of the particular limits. A practical consideration is that the new variable $u$ be a known function of $x$: $[u = u(x)]$.

\[ \int_{x_{\text{lower}}}^{x_{\text{upper}}} f(x) \, dx = \int_{u_{\text{lower}}}^{u_{\text{upper}}} g(u) \, du \quad \text{where} \quad du = \frac{d[u(x)]}{dx} \, dx \]

\[ \int_{x_{\text{lower}}}^{x_{\text{upper}}} f(x) \, dx = \int_{u_{\text{lower}}}^{u_{\text{upper}}} \left\{ f(x) \left[ \frac{du}{dx} \right]^{-1} \right\} \left( \frac{du}{dx} \right) dx = \int_{u_{\text{lower}}}^{u_{\text{upper}}} g(u) \, du \]
The integrand is $g(u) = \left[ f(x) \left( \frac{du}{dx} \right)^{-1} \right]_{for \ x \ x(x) = u}$, and the new differential is $du = \left( \frac{du}{dx} \right)dx$. The conclusion is that $g(u)$ is found by removing a factor of $\frac{du}{dx}$ from $f(x)$ and then expressing the remaining factor as a function of $u$. The new differential $du$ is equal to the factor removed $\frac{du}{dx}$ times the differential of the original integration variable $dx$. The limits of the new variable must correspond to those of the old variable. The new lower limit is the value of $u$ that corresponds to the original lower limit $u(x_{lower})$. Similarly, the new upper limit is $u(x_{upper})$.

\[
\sum_{i=1}^{N} f(x_i) \Delta x_i = \sum_{i=1}^{N} f(x_i) \frac{\Delta x_i}{\Delta u_i} \Delta u_i = \sum_{i=1}^{N} \left\{ f(x_i) \left[ \frac{\Delta u_i}{\Delta x_i} \right]^{-1} \right\} \Delta u_i = \sum_{i=1}^{N} g(u_i) \Delta u_i
\]

The integrals, areas under the curves, are to be the same. For the first shaded blocks, the width $\Delta u$ in the right-hand plot appears to be less than the width $\Delta x$ of the corresponding shaded block in the left-hand plot. The widths scale as $\Delta u = \left( \frac{u_2-u_1}{x_2-x_1} \right) \Delta x$ so the average values of the functions in those intervals must scale as $g_{ave} = \left( \frac{x_2-x_1}{u_2-u_1} \right) f_{ave}$ to ensure that $f_{ave} \Delta x = g_{ave} \Delta u$. As the areas for the complete integrals are shaded, it becomes evident that $u_{limit} = u(x_{limit})$. In the limit that the widths of the blocks approaches zero, $\left( \frac{u_2-u_1}{x_2-x_1} \right) \frac{du}{dx}$, so the new differential and function are $du = \left( \frac{du}{dx} \right)dx$ and $g(u) = \left[ f(x) \left( \frac{du}{dx} \right)^{-1} \right]_{for \ x \ x(x) = u}$.

**Change of variable – a practical example:**
\[ \int_1^2 e^t \sin(e^t + 2) \, dt \]. The new variable is the argument of the most complicated function in the integrand. \( u = e^t + 2; \, du = e^t \, dt \). Solve for \( dt = e^{-t} \, du \). Substitute:

\[ \int e^t \sin(e^t + 2) \, dt \rightarrow \int e^t \sin[u](e^{-t} \, du) = \int \sin[u] \, du = -\cos[u] \]

Reverse the substitution:

\[ \int e^t \sin(e^t + 2) \, dt \rightarrow -\cos[e^t + 2] \]

As the result is now expressed in terms of the original variables, the original limits apply.

Alternate evaluation: Change the limits each time you make a substitution.

\[ \int_1^2 e^t \sin(e^t + 2) \, dt \rightarrow \int_{e^1 + 2}^{e^2 + 2} e^{-t} \sin[u] \, du = \int_{e^1 + 2}^{e^2 + 2} \sin[u] \, du = \left[-\cos[u]\right]_{e^1 + 2}^{e^2 + 2} = -\cos[e^2 + 2] + \cos[e^1 + 2] \]

The second evaluation has the advantage that (with limits) the expressions are complete at each step.

**Trigonometric substitutions:** Indicated if change of variable has failed and if the sum or difference of squares is present (or better yet, the square root of the sum or difference of squares).

\[ \sqrt{a^2 + x^2} \Rightarrow \tan \theta = \frac{x}{a} \text{ as } 1 + \tan^2 \theta = \sec^2 \theta \text{ and } d(\tan \theta) = \sec^2 \theta \, d\theta \]

\[ \sqrt{a^2 - x^2} \Rightarrow \sin \theta = \frac{x}{a} \text{ as } 1 - \sin^2 \theta = \cos^2 \theta \text{ and } d(\sin \theta) = \cos \theta \, d\theta \]

and sometimes forms like \( \sqrt{1 - \left(\frac{x}{b}\right)^2} \) can be attacked using \( \sin^2 \theta = \frac{b^2}{a^2} \)

The trigonometric identities suggest a \( \tan \theta \) be used for the sum of squares and a \( \sin \theta \) be used for the difference of squares. Be aware that the angle \( \theta \) chosen as the new variable is meaningful. Identify \( \theta \) on your figure. Interpret your results in terms of this angle if possible. Square roots are often used to represent distances in physics. *If this is the case, only the positive root is meaningful.* Be alert and examine cases. The functions \( \sin \theta \) and \( \tan \theta \) are preferred choices because they are monotone, increasing functions for domains in the interval \((-\frac{1}{2}, \frac{1}{2})\). Also the ranges of \( \sin \theta \) and \( \tan \theta \) are appropriate for \( \frac{x}{a} \) in the expressions \( \sqrt{1 - \left(\frac{x}{a}\right)^2} \) and \( \sqrt{1 + \left(\frac{x}{a}\right)^2} \).

**Desperation substitution alternatives:** the hyperbolic functions. The trigonometric substitutions can fail to lead to an easily integrated form. Hyperbolic function substitutions provide a remedy in
some cases. ** See sample calculation IB.9 for an evaluation of the inverse hyperbolic functions in term of logarithms.

\[
\sqrt{a^2 + x^2} \Rightarrow \sinh(u) = \frac{x}{a} \quad \text{as} \quad 1 + (\sinh(u))^2 = (\cosh(u))^2 \quad \text{and} \quad d(\sinh(u)) = \cosh(u) \, du
\]

\[
\sqrt{a^2 - x^2} \Rightarrow \tanh(u) = \frac{x}{a} \quad \text{as} \quad 1 - (\tanh(u))^2 = (\sech(u))^2 \quad \text{and} \quad d(\tanh(u)) = (\sech(u))^2 \, du
\]

\[
\sqrt{x^2 - a^2} \Rightarrow \cosh(u) = \frac{x}{a} \quad \text{as} \quad (\cosh(u))^2 - 1 = (\sinh(u))^2 \quad \text{and} \quad d(\cosh(u)) = \sinh(u) \, du
\]

**Trig/Hyper: Substitution Table:**

<table>
<thead>
<tr>
<th>Sign</th>
<th>Range</th>
<th>( \sqrt{a} ) Choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>(</td>
<td>\sqrt{a}</td>
</tr>
<tr>
<td>+</td>
<td>(</td>
<td>\sqrt{a}</td>
</tr>
<tr>
<td>-</td>
<td>(</td>
<td>\sqrt{a}</td>
</tr>
<tr>
<td>+</td>
<td>(</td>
<td>\sqrt{a}</td>
</tr>
</tbody>
</table>

For \( a \pm x \to a \left(1 + [\sqrt{a}]^2\right) \), you might try \( \sqrt{a} \to \sin^2 u \), etc. particularly if \( x^{3/2} \) appears elsewhere.

**Integration by parts:**

\[
\frac{d(uv)}{dx} = u \frac{dv}{dx} + \frac{du}{dx} v \quad \text{or} \quad u \frac{dv}{dx} = \frac{d(uv)}{dx} - \frac{du}{dx} v.
\]

In terms of differentials: \( d(uv) = u \, dv + v \, du \) or \( u \, dv = d(uv) - v \, du \)

\[
\int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du
\]

A common case is one the form \( \int_a^b \frac{\partial (f(x,t))}{\partial x} g(x,t) \, dx \). In this case, \( u = g(x,t) \) and \( dv = \frac{\partial f(x,t)}{\partial x} \, dx \). It follows that:

\[
\int_a^b \frac{\partial (f(x,t))}{\partial x} g(x,t) \, dx = \left[ f(x,t) g(x,t) \right]_{x=a}^{x=b} - \int_a^b f(x,t) \left( \frac{\partial (g(x,t))}{\partial x} \right) \, dx
\]

The integration by parts yields the surface term, \( \left[ f(x,t) g(x,t) \right]_{x=a}^{x=b} \), plus the result of moving the derivative from one factor of the integrand to the remaining factor with a change of sign.

**SPECIAL CASE:** In many integration by parts problems, particularly those arising in quantum mechanics, the surface term vanishes \( \Rightarrow \)

\[
\int_a^b \frac{\partial (f(x,t))}{\partial x} g(x,t) \, dx = -\int_a^b f(x,t) \left( \frac{\partial (g(x,t))}{\partial x} \right) \, dx
\]

The overall action is to move the derivative with respect to the integration variable from one factor of the integrand to the remaining factor
with a change of sign in that special case.

**Applying Integration by Parts:** \( \int x^n e^{-x} \, dx \). The \( n = 0 \) case is easy so the by parts method is applied to lower \( n \). Choose \( v = x^n \) and \( du = e^x \, dx \).

\[
\int x^n e^{-x} \, dx = x^n (-e^{-x}) - \int n x^{n-1} (-e^{-x}) \, dx
\]

**Tip 1:** If multiple integration by parts cycles are necessary, be sure to keep applying the method in the same sense. Here one chooses \( v \) to be the power of \( x \) so the power is lowered in every step. Reversing the sense would just undo a previous step.

**Tip 2:** After each cycle consider reshuffling factors between the \( u \) and \( v \) factors.

\[
\int x \sin^{-1}(x) \, dx = \frac{1}{2} x^2 \frac{1}{\sqrt{1-x^2}} - \frac{1}{2} \int x^2 \frac{1}{\sqrt{1-x^2}} \, dx
\]

\[
\downarrow
\]

\[
= \frac{1}{2} x^2 \frac{1}{\sqrt{1-x^2}} - \frac{1}{2} \int (x) \left( \frac{x}{\sqrt{1-x^2}} \right) \, dx
\]

**Tip 3:** In some cases the integration by parts method generates a multiple (*other than 1*) of the original integral on the right hand side.

\[
\int e^{-x} \sin(kx) \, dx = -e^{-x} \sin(kx) + k \int e^{-x} \cos(kx) \, dx
\]

\[
= -e^{-x} \sin(kx) - k e^{-x} \cos(kx) - k^2 \int e^{-x} \sin(kx) \, dx
\]

\[
\int e^{-x} \sin(kx) \, dx = \frac{-e^{-x} \sin(kx) - k e^{-x} \cos(kx)}{1 + k^2}
\]

**Tip 4:** Get familiar with derivatives and anti-derivatives of factors and identities: \( \int \sec^3(x) \, dx \).

It is not obvious that one should try integration by parts. If one recalls that:

\[
\frac{d \tan(x)}{dx} = \sec^2(x); \quad \frac{d \sec(x)}{dx} = \sec(x) \tan(x); \quad \sec^2(x) = 1 + \tan^2(x)
\]

then one might try \( u = \sec(x) \) and \( dv = \sec^2(x) \, dx \). This choice in examined further in problem 47.

**Hints from Steve Kifowot:**
Choosing $u$ and $v$. Choose $u$ that is easy to differentiate and $v$ that is easy to integrate. When the integrand can be factored into logarithmic, inverse trigonometric, polynomial, trigonometric and exponential pieces, choose $u$ to be of the form that appears earliest in the list. Employ the tabular method to organize you work in cases in which the integration by parts must be repeated several times. The method is to be demonstrated for $\int x^3 \cos(x) \, dx$. The factor $x^3$ is polynomial so it is chosen to be $u$.

<table>
<thead>
<tr>
<th>Step + to -</th>
<th>$u$ Step down $\Rightarrow \frac{d}{dx}$</th>
<th>$v$ Step down $\int , dv$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>$x^3$</td>
<td>$\cos(x)$</td>
</tr>
<tr>
<td>-</td>
<td>$3x^2$</td>
<td>$\sin(x)$</td>
</tr>
<tr>
<td>+</td>
<td>$6x$</td>
<td>$- \cos(x)$</td>
</tr>
<tr>
<td>-</td>
<td>$6$</td>
<td>$- \sin(x)$</td>
</tr>
<tr>
<td>+</td>
<td>$0$</td>
<td>$\cos(x)$</td>
</tr>
</tbody>
</table>

$\int x^3 \cos(x) \, dx = x^3 \sin x + 3x^2 \cos(x) - 6x \sin x - 6 \cos x + C$

**Exercise:** Use the tabular method to evaluate $\int x^4 \sin(2x) \, dx$

**Exercise:** Consider the integral $\int x^4 \ln(x) \, dx$. Which factor should be chosen as $u$ and which as $v$?

Execute the tabular procedure. Comment on a possible shortcut.

**Trigonometric Identities:** (This section is to be expanded.)

Many trigonometric identities provide beneficial recasting of integrands. The particular cases presented here are intended to be examples that suggest paths to explore.

**Attack 1**: $\sin \theta$ or $\cos \theta$ raised to an odd power.

Consider $(\sin \theta)^{2n+1}$ as $[1 - (\cos \theta)^2]^n \sin \theta$ which becomes: $- [1 - u^2]^n \, du$.

**Attack 2**: $\sin \theta$ or $\cos \theta$ raised to an even power.

Use the double angle relations repeatedly. Represent $(\sin \theta)^{2n}$ as $[(1 - \cos^2 \theta)/2]^n$ working toward a form with only first powers or trigonometric functions.

**Attack 3**: mixed products of $\sin \phi$, $\sin \alpha$, $\cos \beta$ and $\cos \theta$
Apply the product identity repeatedly to reduce the expression to terms that are first order in trigonometric functions.

\[
\sin(x) \sin(y) = \frac{1}{2} \left[ \cos(x - y) - \cos(x + y) \right]
\]

\[
\cos(x) \cos(y) = \frac{1}{2} \left[ \cos(x - y) + \cos(x + y) \right]
\]

\[
\sin(x) \cos(y) = \frac{1}{2} \left[ \sin(x - y) + \sin(x + y) \right]
\]

These methods can be combined with other methods such as integration by parts to compute a variety of integrals. For example, \( \int x^2 \sin^2(kx) \, dx \) is to be evaluated as one of the end-of-section problems.

**Completing the square:** Several of the previous techniques can be extended by completing a square. For example, the result \( \int \frac{du}{1+u^2} \) can be extended to:

\[
\int \frac{dx}{a x^2 + b x + c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \left( \frac{bx + a}{\sqrt{4ac-b^2}} \right)
\]

Begin by completing the square in the denominator,

\[
\int \frac{dx}{a x^2 + b x + c} = \int \frac{dx}{\left( \sqrt{a} x + \frac{b}{2\sqrt{a}} \right)^2 + \left( c - \frac{b^2}{4a} \right)} = \frac{1}{c - \frac{b^2}{4a}} \int \frac{dx}{1 + \left( \frac{\sqrt{a} x + \frac{b}{2\sqrt{a}}}{\sqrt{c - \frac{b^2}{4a}}} \right)^2}
\]

Next, make the change of variable \( u = \left( \sqrt{a} x + \frac{b}{2\sqrt{a}} \right) / \sqrt{c - \frac{b^2}{4a}} \), \( du = \sqrt{a} \, dx / \sqrt{c - \frac{b^2}{4a}} \). This change is chosen to cast the denominator in the form \( 1 + u^2 \).

\[
\int \frac{dx}{a x^2 + b x + c} = \int \frac{du}{1+u^2} = \frac{\sqrt{c - \frac{b^2}{4a}}}{\sqrt{a} \left( c - \frac{b^2}{4a} \right)} \int \frac{du}{1+u^2} = \frac{2}{\sqrt{4ac-b^2}} \int \frac{du}{1+u^2}
\]

\[
\int \frac{dx}{a x^2 + b x + c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \left( \frac{2ax+b}{\sqrt{4ac-b^2}} \right)
\]

**SampleCalc:**

\[
\frac{1}{x^2 - 2x - 3} = \frac{1}{(x-1)^2 - 4} = \frac{1}{4 \left( (x-1)/2 \right)^2 - 1} = \frac{1}{4u^2 - 1}, \text{ where } u = \frac{1}{2}(x - 1).
\]
**Partial Fractions:** Integrands with complicated denominators present special problems, and the partial fraction approach is an effective method to reduce that complexity when the denominator is an $n^{th}$ order polynomial $q(x)$. In general, we consider integrals of the form:

$$\int \frac{s(x) \, dx}{q(x)} = \int \frac{m(x) \, dx}{q(x)} + \int \frac{p(x) \, dx}{q(x)}$$

Where $s(x) = m(x) \, q(x) + p(x)$ and $p(x)$ is a polynomial of lower order than $q(x)$. That is: $s(x)$ divided by $q(x)$ is $m(x)$ with remainder term $p(x)$. The details of the method and some computational shortcuts are presented in the Tools of the Trade section of this handout. A simple example is all that is presented here. Assume that $q(x)$ has been scaled and that the roots of $q(x) = 0$ are known so that

$$q(x) = x^3 + (2a + b) \, x^2 + (a^2 + 2 \, a \, b) \, x + a^2 \, b = (x - a)^2 \, (x - b).$$

$$\int \frac{dx}{q(x)} = \int \frac{dx}{(x - a)^2 (x - b)}$$

The core of the method is to break the integrand into partial fractions.

$$\frac{1}{(x-a)^2 (x-b)} = \frac{A}{(x-b)} + \frac{Bx + C}{(x-a)^2} = \alpha \frac{1}{(x-b)} + \beta \frac{1}{(x-a)} + \gamma \frac{1}{(x-a)^2}$$

The numerator of each term on the right is an arbitrary polynomial or order $n - 1$ where $n$ is the degeneracy of the root (the order of the polynomial denominator in that term. As an alternative, one can use the ‘all powers up to’ representation in which all the Greek coefficients are scalar constants.

$$1 = \frac{A \, (x-a)^2 \, (x-b)}{(x-b)} + \frac{(Bx + C)(x-a)^2 \, (x-b)}{(x-a)^2} = A \, (x-a)^2 + (Bx + C)(x-b)$$

**Approach 1:** In this approach, one matches the coefficients of each power of $x$ to find $A$, $B$ and $C$. A shortcut follows if $x$ is set to $b$ which yields $A = (b - a)^2$. Next one matches the coefficients of $x^2$ leading to the equations $0 = A + B$ so $B = -(b - a)^2$. Next, the coefficients of $x^1$ matched.

$$0 = -2 \, a \, A - b \, B + C \quad \text{or} \quad C = \frac{2a-b}{(b-a)^2}.$$
\[ \int \frac{dx}{q(x)} = \int \frac{dx}{(x-a)^2(x-b)} = \int \left[ \frac{A\,dx}{(x-b)} + \frac{(B\,x+C)\,dx}{(x-a)^2} \right] \]
\[ = \frac{1}{(b-a)^2} \int \frac{dx}{(x-b)} - \frac{1}{(b-a)^2} \int \frac{dx}{x-a} \int (x+b-2a)\,dx \]

Using the change of variable \( u = x-a \),
\[ = \frac{1}{(b-a)^2} \int \frac{dx}{(x-b)} - \frac{1}{(b-a)^2} \int \frac{(u+b-a)\,dx}{(u)^2} = \frac{1}{(b-a)^2} \ln(x-b) - \ln(u) + \frac{1}{(b-a)u} \]
\[ \equiv \int \frac{(u+b-a)\,dx}{(u)^2} = \frac{1}{(b-a)^2} \ln(x-b) - \ln(x-a) + \frac{1}{(b-a)(x-a)} \]

Note that these results have not been checked.

**Approach 2:** The process is repeated using the *all powers* form.

\[ \frac{1}{(x-a)^2(x-b)} = \frac{\alpha}{(x-b)} + \frac{\beta}{(x-a)} + \frac{\gamma}{(x-a)^2} \Rightarrow 1 = \alpha(x-a)^2 + \beta(x-a)(x-b) + \gamma(x-b) \]

Setting \( x = a, \gamma = (a-b)^{-1} \). Setting \( x = b, \alpha = (b-a)^{-2} \). The final coefficient follows by setting the coefficient of \( x^2 = 0 \) on both sides \( \Rightarrow \beta = -\alpha = - (b-a)^{-2} \).

(Note: There are many methods to solve the simultaneous equations to find \( A \) and \( B \). The method just presented is just one of the possible ones.)

\[ \int \frac{dx}{q(x)} = \frac{1}{(b-a)^2} \int \frac{dx}{(x-b)} - \frac{1}{(b-a)^2} \int \frac{dx}{(x-a)} + \frac{1}{(a-b)} \int \frac{dx}{(x-a)^2} \]
\[ \int \frac{dx}{q(x)} = \int \frac{dx}{(x-a)^2(x-b)} = \frac{1}{(b-a)^2} \left[ \ln(x-b) - \ln(x-a) \right] + \frac{-1}{(a-b)(x-a)} \]

**The Method of Partial Fraction is effective for integrands that are rational functions,** the ratio of a polynomial numerator to a polynomial denominator in which the numerator is lower order than the denominator. For more details, see Sample Calculation IB.10 and the Tools of the Trade section.

**SCx: Partial Fractions**
\[
\int \frac{x^4 - 4x^3 + 2x^2 - 3x + 1}{(x^2 + 1)^3} \, dx
\]

\[
x^4 - 4x^3 + 2x^2 - 3x + 1 = x^2 - 4x + 1 \, \text{remainder}, x
\]

\[
x^4 - 4x^3 + 2x^2 - 3x + 1 = \frac{x^2 - 4x + 1}{(x^2 + 1)^2} + \frac{x}{(x^2 + 1)^3}
\]

Repeat the process:

\[
x^2 - 4x + 1 \quad \text{remainder}, 4x
\]

\[
x^4 - 4x^3 + 2x^2 - 3x + 1 = \frac{1}{(x^2 + 1)^2} + \frac{4x}{(x^2 + 1)^3}
\]

which yields the reduced form:

\[
\int \frac{x^4 - 4x^3 + 2x^2 - 3x + 1}{(x^2 + 1)^3} \, dx = \int \frac{1}{(x^2 + 1)^2} + \frac{4x}{(x^2 + 1)^3} \, dx
\]

**Exercise:** Show that \( \int \frac{x}{(x^2 + 1)^n} \, dx = \frac{-1}{2(n-1)} (x^2 + 1)^{-n+1} \) and that \( \int \frac{dx}{(x^2 + 1)^n} = \int (\cos \theta)^{2n-2} \, d\theta \) where \( \theta = \tan^{-1}(x) \).

**Parameter Calculus**\(^1\): There are cases in which the integrand can be defined to depend on one or more parameters in addition to the integration variable. It may be possible to integrate (or differentiate) with respect to a parameter to change the integrand to a form that can be more easily integrated. After the integration, the result is then differentiated (or integrated) with respect to that same parameter to recover the value of the original integral. Suppose \( \int x \cos(x) \, dx \) is desired.

\[
\int \sin(bx) \, dx = -\frac{\cos(bx)}{b} \quad \text{and} \quad \frac{d}{db} \int \sin(bx) \, dx = \int x \cos(bx) \, dx
\]

so:

\[
\int x \cos(bx) \, dx = \frac{d}{db} \int \sin(bx) \, dx = \frac{d}{db} \left( -\frac{\cos(bx)}{b} \right) = \frac{x \sin(bx)}{b} + \frac{\cos(bx)}{b^2}
\]

\(^1\) The parameter calculus technique is sometimes called *Feynman’s integration trick.*
Setting $b = 1$:

$$\int x \cos(x) \, dx = x \sin(x) + \cos(x)$$

The taking a derivative with respect to a parameter trick is demonstrated more fully in the definite integrals handout. Sample Calculation IB.8 demonstrates a more advanced application of the parameter calculus technique.

* The order of integrations and differentiations are interchanged freely in the parameter calculus approach. This freedom is only justified if strong (uniform) convergence of the operations can be established. Good fortune is to be assumed in this section. Uniform convergence is to be discussed later in a separate handout.

**Parameter Calculus** is used extensively in several fields. Also, integrals are sums so the technique applies to sums as well. A standard trick used in statistical mechanics is to take a derivative with respect to a parameter in order to insert a factor of interest into an average. For a binomial distribution, we need to insert a factor of $n_1$ into $1 = \sum_{n=0}^{N} \frac{N!}{n_1!(N-n_1)!} [p^n q^{(N-n_1)}]$ to find $\bar{n}_1 = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} [p^n q^{(N-n_1)}] n_1$. We start by taking a partial derivative with respect to $p$. Note that $p$ is a **parameter of the problem**, not a variable. Nonetheless, the formal derivative of $\left[p^n q^{(N-n)}\right]$ with respect to $p$ yields a factor of $n_1$ at the cost of dropping the power of $p$.

$$\frac{\partial}{\partial p} \left[p^n q^{(N-n_1)}\right] = n_1 p^{(n_1-1)} q^{(N-n_1)}$$

We next patch the relation by multiplying by $p$.

$$p \frac{\partial}{\partial p} \left[p^n q^{(N-n_1)}\right] = p n_1 p^{(n_1-1)} q^{(N-n_1)} = \left[p^n q^{(N-n_1)}\right] n_1$$

It follows that $\bar{n}_1 = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} \left[p^n q^{(N-n_1)}\right] n_1 = p \frac{\partial}{\partial p} \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} \left[p^n q^{(N-n_1)}\right] = p \frac{\partial}{\partial p} [p+q]^N = Np[p+q]^{N-1} = Np = \bar{n}_1$ as $p + q = 1$.

Work through the calculation of $\bar{n}_1^2$. Start with $\bar{n}_1^2 = \sum_{n_1=0}^{N} \frac{N!}{n_1!(N-n_1)!} \left[p^n q^{(N-n_1)}\right] n_1^2 = p \frac{\partial}{\partial p} \left(p \frac{\partial}{\partial p} [p+q]^N\right) = p \frac{\partial}{\partial p} [p+q] N^2 p^2 + Npq = (\bar{n}_1)^2 + Npq$
Alternate Representations: Use identities to recast the form of the integrand. Standard substitutions include using the Euler identity to convert trigonometric functions to exponentials. This path is almost always indicated if there is another exponential factor in the integrand. The hyperbolic functions also have exponential definitions. An integrand may, for example, contain a lonely cosine, one that is lacking a sine for its ‘du’. The replacement $2 + 2 \cos(2 \phi) = \cos^2 \phi$ might be followed by division by $\cos^2 \phi$ and the substitution $\sec^2 \phi = \tan^2 \phi + 1$.

Tricks presented by your instructor in the course. Each topic in physics has its own set of standard integrals and integration techniques. As you prepare course summary sheets for each course that you take, add a page listing the math skills required for that course. Prepare a math methods summary for all the techniques that you use that includes the various applications of each technique. It often follows that if similar math methods are used in two fields of physics that those fields are more closely related than might be first evident. The math parallels may lead you to a deeper appreciation of the physics parallels. Parameter calculus is often considered to lie in the trick category.

Integral tables or Mathematica, but only as a last resort or a check. The mathematics represents the physics. Only by stepping through the solutions to the mathematics can you step through the linkages in the physics.

Once you complete the calculation, reflect on your efforts. Review the techniques that were successful and attempt to identify clues that would lead you to select them more quickly. Attempt to re-express your results in the language of the problem statement and without any direct reference to the particular coordinate system that was used. For example: The electric field due to a long straight uniformly charged wire varies as the inverse of the distance from the wire, and it is directed perpendicularly away from the wire at any point. This final step helps you extract the physics content of the problem.

Sample Calculation IB.1: Change of Variable for $\int_{2}^{4} \frac{x \, dx}{1 + x^2}$. The guideline suggests that the argument of the most troublesome functional form in the integrand be chosen as the new variable.
The argument of the inverse power is \( u = 1 + x^2 \). The next step to compute \( du = \frac{du}{dx} \, dx = 2 \, x \, dx \).

This outcome is the good one; \( du \) can be formed as \( x \, dx = \frac{1}{2} \, du \). As \( u = 1 + x^2 \), \( u_{\text{lower}} = u(\text{for } x = 2) = 5 \), and \( u_{\text{upper}} = u(\text{for } x = 4) = 17 \).

\[
\int_{\frac{1}{2}}^{4} \frac{x \, dx}{1 + x^2} = \int_{u \text{ for } x = 2}^{u \text{ for } x = 2} \frac{u}{2} \, du \rightarrow \int_{5}^{17} \left( \frac{1}{2} \right) \, \frac{du}{u} = \left( \frac{1}{2} \right) \ln \left( \frac{17}{5} \right) \approx 0.6119
\]

A crucial point is the limits of integration must be changed whenever the variable is changed. It is rewarding that, after changing variables, the actual integral calculated is simple.

**Sample Calculation IB.2:** Change of Variable for \( \int_{0}^{\pi} \frac{(R-r \cos \theta) \sin \theta \, d\theta}{[R^2 + r^2 - 2R \cos \theta]^{(3/2)}} \). The guideline suggests that the argument of the most troublesome functional form in the integrand be chosen as the new variable. That is, the argument of \( [ \ldots ]^{3/2} \); so

\[
u = R^2 + r^2 - 2R \cos \theta \text{ and hence } du = 2 \, R \, r \, \sin \theta \, d\theta
\]

The process has some chance to succeed as \( du \) can be found. (\( R \) and \( r \) are positive constants that represent characteristic lengths.)

\[
\sin \theta \, d\theta = \frac{du}{2R \, r}
\]

The limits transform by evaluating the new integration variables as functions of the original limits as

\[
u(\theta = 0) = R^2 + r^2 - 2R = (R - r)^2 \text{ and } u(\theta = \pi) = (R + r)^2 \text{ so we have:}
\]

\[
\left( \frac{\sqrt{2} \, R \, r}{u^{3/2}} \right) \int_{(R-r)^2}^{(R+r)^2} \frac{(R-r \cos \theta) \, du}{u^{3/2}}
\]

The remaining pieces must be expressed in terms of the new variable. The \( R \) is no problem, as it is just a constant. Attack the term \( r \, \cos \theta \). It appears in the definition of \( u \). \( \Rightarrow R \, \cos \theta = \frac{R^2 + r^2 - u}{2R} \) and

\[
R - r \, \cos \theta = \frac{R^2 - r^2 + u}{2R}.
\]

\[
\left( \frac{\sqrt{2} \, R \, r}{u^{3/2}} \right) \int_{(R-r)^2}^{(R+r)^2} \left[ \frac{(R^2 - r^2) \, u^{-3/2} + u^{-1/2}}{R^2 + r^2 - 2R \cos \theta} \right] \, du = \left( \frac{\sqrt{2} \, R \, r}{u^{3/2}} \right) \left[ -2(R^2 - r^2) \, u^{-1/2} \left( \frac{(R+r)^2}{(R-r)^2} + 2u^{1/2} \right) \left( \frac{(R-r)^2}{(R+r)^2} \right) \right]
\]

\[
\int_{0}^{\pi} \frac{(R-r \cos \theta) \sin \theta \, d\theta}{[R^2 + r^2 - 2R \cos \theta]^{(3/2)}} = \left( \frac{\sqrt{2} \, R \, r}{u^{3/2}} \right) \left[ \frac{(R^2 - r^2)}{(R-r)^2} - \frac{(R^2 - r^2)}{(R+r)^2} + \sqrt{(R + r)^2} - \sqrt{(R - r)^2} \right]
\]
Assuming that the square roots arose as representations of distances, positive roots are to be taken.

\[ \sqrt{(R-r)^2} \rightarrow |R-r| \]

\[ \int_0^\pi \frac{(R-r \cos \theta) \sin \theta d\theta}{R^2 + r^2 - 2rR \cos \theta}^{(3/2)} = \left( \sqrt[2]{2} r R \right) \left[ \frac{(R-r)(R+r)-(R-r)+ (R+r)-|R-r|}{|R-r|} \right] \]

Note that the final integral has two distinct analytic forms: one for \( r < R \) and another for \( r > R \).

**Sample Calculation IB.3:** Trigonometric Substitution. A calculation of the electric field at a point \( \hat{j} \) due to a uniformly charged line running from \( x_1 \) to \( x_2 \) along the \( x \) axis leads to the integrals.

\[ \frac{\lambda}{4\pi\varepsilon_0} \left\{ \left[ \int_{x_1}^{x_2} \frac{x \ dx}{\left[ s^2 + x^2 \right]^{3/2}} \right] \hat{i} + \left[ \int_{x_1}^{x_2} \frac{s \ dx}{\left[ s^2 + x^2 \right]^{3/2}} \right] \hat{j} \right\} \]

The first integral surrenders to the change of variable approach, but the second resists the technique. If one chooses \( s^2 + x^2 \) as the new variable \( u \), then \( du = 2x \ dx \), a factor that cannot be found. If \( s \ dx \) is written as \( s \ x^{-1} (1/2 \ du) \), one is left the task of expressing \( x^{-1} \) as a function of \( u \) \( (x = \pm \sqrt{u-s^2}) \). Life is not getting simpler. By rule, a trigonometric substitution is to be tried next. The sum of squares form suggests a \( \tan \theta \) substitution. Guided by the form of the identity \( \sec^2 \theta = 1 + \tan^2 \theta \), the integral is recast as:

\[ \int_{x_1}^{x_2} \frac{s \ dx}{\left[ s^2 + x^2 \right]^{3/2}} = \frac{1}{s^3} \int_{\sin \theta_1}^{\sin \theta_2} \frac{dx_2}{\left[ 1 + (x_2/s)^2 \right]^{3/2}} = \frac{1}{s} \int_{\tan \theta_1}^{\tan \theta_2} \frac{d(\tan \theta)}{\left[ 1 + (\tan \theta)^2 \right]^{3/2}} = \frac{1}{s} \int_{\theta_1}^{\theta_2} \sec^2 \theta \ d\theta \]

where \( \tan \theta = x/s \) or \( x = s \tan \theta \) and \( dx = s \ d(\tan \theta) = s \sec^2 \theta \). After all the sweat, the integral reduces to:

\[ \frac{1}{s} \int_{\theta_1}^{\theta_2} \sec^2 \theta \ d\theta = \frac{1}{s} \int_{\theta_1}^{\theta_2} \cos \theta \ d\theta = \frac{1}{s} \left[ \sin \theta_2 - \sin \theta_1 \right] \]

The conversion from \( \theta \) values to \( x \) values is facilitated by sketching a triangle with a side of length \( x \) opposite to the angle \( \theta \) and a side of length \( s \) adjacent to it. The conversions can be read off the figure for the triangle. It follows that:

\[ \sin \theta = \frac{x}{\sqrt{s^2 + x^2}} \quad \text{and} \quad \cos \theta = \frac{s}{\sqrt{s^2 + x^2}} \]

leading to the integrals.
\[
\int_{y_1}^{y_2} x \, dx \left[ \frac{1}{s^2 + x^2} \right]^{3/2} = \left[ -\frac{1}{\sqrt{s^2 + x^2}} + \frac{1}{\sqrt{s^2 + y_1^2}} \right] \quad \text{and} \quad \int_{y_1}^{y_2} s \, dx \left[ \frac{1}{s^2 + x^2} \right]^{3/2} = \frac{1}{s} \left[ -\frac{x_2}{\sqrt{s^2 + x_2^2}} - \frac{x_1}{\sqrt{s^2 + y_1^2}} \right]
\]

**Exercise**: Complete the evaluation of the two integrals.

**Sample Calculation IB.4**: Integration by Parts of: \( \int f(x) \, dx \). (Let \( u = f(x) \) and \( dv = dx \))

\[
\int f(x) \, dx = x \cdot f(x) - \int x \left( \frac{df}{dx} \right) \, dx
\]

For example, let \( f(x) = \ln(x) \), and hence \( \frac{df}{dx} = \frac{1}{x} \).

\[
\int \ln(x) \, dx = x \cdot \ln(x) - \int x \left( \frac{1}{x} \right) \, dx = x \ln(x) - x
\]

**Sample Calculation IB.5**: Integration by Parts.

\[
\int_a^b u \, dv = uv \bigg|_a^b - \int_a^b v \, du
\]

Compute \( \int_0^3 t^2 \, e^{-t} \, dt \). Use \( dv = e^{-t} \, dt \) and hence \( v = -e^{-t} \). Do it twice. The first time, \( u = t^2 \) so \( du = 2 \, t \, dt \).

\[
\int_0^3 t^2 \, e^{-t} \, dt = t^2 \left( -e^{-t} \right) \bigg|_0^3 - \int_0^3 2 \left( t \, e^{-t} \right) \, dt = -9 e^{-3} + 2 \left( -e^{-t} \right) \bigg|_0^3 2 \left( -e^{-t} \right) \, dt = -9 e^{-3} - 6 e^{-3} - 2 e^{-3} + 2
\]

**Sample Calculation IB.6**: Parameter Calculus. The previous integration is to be repeated using parameter techniques. The procedure is anchored by:

\[
\int_0^3 e^{-bt} \, dt = \left( \frac{1}{b} \right) (1 - e^{-3b})
\]

Noting that a derivative with respect to \( b \) brings down a factor of minus \( t \),

\[
\frac{d^2}{db^2} \left( \int_0^3 e^{-bt} \, dt \right) = \frac{d^2}{db^2} \left[ \left( \frac{1}{b} \right) (1 - e^{-3b}) \right]
\]

\[
= \frac{d}{db} \left[ \left( -\frac{1}{b^2} \right) (1 - e^{-3b}) + 3 \left( \frac{1}{b} \right) e^{-3b} \right] = \left( \frac{2}{b^3} \right) (1 - e^{-3b}) + \left( -\frac{3}{b^2} \right) e^{-3b} + \left( -\frac{3}{b^2} \right) e^{-3b} - \left( \frac{9}{b} \right) e^{-3b}
\]

Setting \( b = 1 \),

\[
\int_0^3 t^2 \, e^{-t} \, dt = 2 - 17 e^{-3}
\]
The final step is the evaluation of the parameter at a special value, often one. It seems like cheating at first, but, in time, the procedure gains credibility.

**Sample Calculation IB.7:** Substituting exponential representations for trig and hyperbolic functions. 

\[ \cos \theta + i \sin \theta = e^{i\theta} \]

\[
\int_0^a e^{-x} \sin(x) \, dx = \int_0^a e^{-x} \left[ \frac{e^{ix} - e^{-ix}}{2i} \right] \, dx = \int_0^a \left[ \frac{e^{-i|x|} - e^{-|i|x|}}{2i} \right] \, dx
\]

\[
\left( \frac{1}{2i} \right) \left[ -\frac{e^{-[1+i]x}}{1-i} + \frac{e^{-[1+i]x}}{1+i} \right]_0^a = \left( \frac{1}{2i} \right) \left[ \frac{1}{1-i} - \frac{1}{1+1} \right] + \left( \frac{1}{2i} \right) \left[ -\frac{e^{-[1-i]a}}{1-i} + \frac{e^{-[1+i]a}}{1+i} \right]
\]

\[
= \left( \frac{1}{2i} \right) \left[ \frac{2i}{1+1} \right] + \left( \frac{e^{-i}a}{2i} \right) \left[ -\frac{e^{-i}a}{1-i} + \frac{e^{-i}a}{1+i} \right] = \left( \frac{1}{2} \right) + \left( \frac{e^{-a}}{2i} \right) \left( \frac{1}{2} \right) \left[ -[e^{-a} - e^{-ia}] - i[e^{ia} + e^{-ia}] \right]
\]

\[ \Rightarrow \int_0^a e^{-x} \sin(x) \, dx = \frac{1}{2} \left[ 1 - e^{-a} \sin(a) - e^{-a} \cos(a) \right] \]

**Sample Calculation IB.8: Parameter Calculus.** In extreme cases, the expression to be evaluated may be shown to satisfy a differential equation with the parameter as the independent variable that is to be solved for the desired value of the parameter given that the value at a boundary (special value of the parameter) can be computed more directly. In the example taken from McQuarrie, a complicating piece of an expression is removed by setting a parameter to zero. The full expression is evaluated by showing that it satisfies a differential equation in that parameter. The DE is solved and boundary value matched at zero.

\[ I(a,b) = \int_0^\infty e^{-ax^2 + bx^2} \, dx \]

The \( b = 0 \) case (\( \int_0^\infty e^{-a x^2} \, dx = \frac{\sqrt{\pi}}{2a} \)) is a standard integral, and its evaluation is presented in the definite integrals handout. The derivative of \( I(a,b) \) is computed next.

\[ \frac{\partial I(a,b)}{\partial b} = \frac{\partial}{\partial b} \int_0^\infty e^{-[a^2 x^2 + b^2 x^2]} \, dx = \int_0^\infty e^{-[a^2 x^2 + b^2 x^2]} \left(-2bx^2\right) \, dx \]

Choosing \( z = (b/a)x^2 \) and hence \( dz = -\left(b/a\right)x^2 \, dx \).

\[ \frac{\partial I(a,b)}{\partial b} = 2a \int_0^\infty e^{-\left[b^2 z^2 + a^2 z^2\right]} \, dz = -2a I(a,b) \]
Collecting the pieces:

\[ \frac{\partial I(a,b)}{\partial b} = -2a I(a,b) \Rightarrow \int_{I(a,b)} \frac{dl}{I} = -2a \int_{0}^{b} db \quad \text{and} \quad I(a,0) = \frac{\sqrt{\pi}}{2a} \Rightarrow I(a,b) = \frac{\sqrt{\pi}}{2a} e^{-2ab} \]

**Sample Calculation IB.9: Hyperbolic Function Substitution:**

\[ \int_{x_1}^{x_2} \frac{dx}{\sqrt{a^2 + x^2}} = \int_{u_1}^{u_2} \frac{\cosh(u)du}{\sqrt{1+(\sinh(u))^2}} = \int_{u_1}^{u_2} du = u_2 - u_1 \]

The substitution simplifies the integration dramatically, but the task of computing the new limits remains. Under the change of variable, sinh(u) = \( \frac{x}{a} \), u = sinh\(^{-1}\)\( (\frac{x}{a}) \). It is most direct to start with the change of variable prescription: sinh(u) = \( \frac{e^u - e^{-u}}{2} \) as 1 + (sinh(u))\(^2\) = (cosh(u))\(^2\) and d(sinh(u)) = cosh(u) du

\[ \sqrt{a^2 - x^2} \Rightarrow \sinh(u) = \frac{x}{a} \quad \text{as} \quad 1 - (\tanh(u))^2 = (\text{sech}(u))^2 \quad \text{and} \quad d(\tanh(u)) = (\text{sech}(u))^2 du \]

\[ \sqrt{x^2 - a^2} \Rightarrow \cosh(u) = \frac{x}{a} \quad \text{as} \quad (\cosh(u))^2 - 1 = (\sinh(u))^2 \quad \text{and} \quad d(\cosh(u)) = \sinh(u) du \]

Based on the templates, the change of variable sinh(u) = \( \frac{x}{a} \) is chosen. So \( x = a \sinh(u) \) and \( dx = a \cosh(u) \ du \).

\[
\int_{x_1}^{x_2} \frac{dx}{\sqrt{a^2 + x^2}} = \int_{u_1}^{u_2} \frac{\cosh(u)du}{\sqrt{1+(\sinh(u))^2}} = \int_{u_1}^{u_2} du = u_2 - u_1
\]

Sample Calculation IB.10: Partial Fractions applied to integrating a rational function:

\[
\int \frac{p(x)}{q(x)} \, dx = \int \frac{(4x+2)}{x^2+3x+2} \, dx = \int \frac{(4x+2)}{(x+1)(x+2)} \, dx = \int \left( \frac{A}{x+1} + \frac{B}{x+2} \right) \, dx
\]

\[
\frac{(4x+2)}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2}
\]

\[
(4x+2) = A(x+1)(x+2) + B(x+1)(x+2)
\]

Setting \( x = -1 \), \((4[-1]+2)= A ([{-1}]+2)\), or \(A = -2\). Setting \( x = -2 \) leads to \( B = 6 \).
\[ \int \frac{4x+2}{x^2+3x+2} \, dx = \int \frac{-2}{x+1} \, dx + \int \frac{6}{x+2} \, dx = 6\ln(x+2) - 2\ln(x+2) \]

Beware: If the denominator has repeated roots, a complication arises. See Tools of the Trade.

The last sample calculation did not display the limits of the integrations. BEWARE: An integral without limits is an integral up to no good.

Tools of the Trade:

**Simple Methods for Times of Desperation:** CAREFUL: Look for df/dx typo below

**Method D1:** Simple integration by parts

\[ \int f(x) \, dx \text{ with } u = f(x) \text{ and } dv = dx \]

\[ \int f(x) \, dx = xf(x) - \int x \frac{f(x)}{dx} \, dx \]

**Method D2:** Simple change of variable

\[ \int f(x) \, dx = \int u \left[ \frac{df}{dx} \right]^{-1} \, du \text{ with } x = f^{-1}[u]; \quad u = f(x), \quad du = \frac{df}{dx} \, dx \]

Let’s try the methods for \( f(x) = \ln(x) \).

**Method D1:** Simple integration by parts

\[ \int \ln(x) \, dx = x\ln(x) - \int x \left( \frac{1}{x} \right) \, dx = x\ln(x) - x \]

**Method D2:** Simple change of variable

\[ \int \ln(x) \, dx = \int u \left[ \frac{df}{dx} \right]^{-1} \, du \text{ with } x = f^{-1}[u]; \quad u = f(x), \quad du = \frac{df}{dx} \, dx \]

\[ \int \ln(x) \, dx = \int u \left[ \frac{1}{e^u} \right]^{-1} \, du = \int u \, x \, du = \int u \, e^u \, du \]

\[ \int \ln(x) \, dx = u \, e^u - \int e^u \, du = [u \, e^u - e^u]_{u=\ln(x)} = x\ln(x) - x \]

**The Role of Patience:** Patience was cheap before the discovery of television. One needed to find something to occupy a rainy afternoon – what better than computing a new integral. The integral
\[ \int \sec(x) \, dx \] is to be computed as a demonstration. Even as the path to the answer is displayed, the two dead-ends taken at each fork in the solution are not. You should beat an integral to death once a month to develop mental toughness.

Staring at the form \[ \int \frac{1}{\cos(x)} \, dx \] is futile so a list of trig identities is \textit{shot-gunned} at the problem. If the integral is to yield, there must be both sines and cosines. A fruitful choice is \[ \cos(x) = \cos^2(\frac{\gamma_2}{2}) - \sin^2(\frac{\gamma_2}{2}). \] The clue that it might work is that a channel opens, a ‘partial fractions’ channel. It utilizes \( x^2 - y^2 = (x+y)(x-y); \) a favorite from high school algebra.

\[ \int \frac{dx}{\cos^2(\frac{\gamma_2}{2}) - \sin^2(\frac{\gamma_2}{2})} = \int \frac{A(x) \, dx}{\cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2})} + \int \frac{B(x) \, dx}{\cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2})} \]

It requires that \( A(x) \left[ \cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2}) \right] + B(x) \left[ \cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2}) \right] = \int \] using the ultimate trig identity \( \cos^2(\frac{\gamma_2}{2}) + \sin^2(\frac{\gamma_2}{2}) = 1. \)

\[ A(x) = -\sin(\frac{\gamma_2}{2}); \quad B(x) = \cos(\frac{\gamma_2}{2}) \Rightarrow \int \frac{dx}{\cos^2(\frac{\gamma_2}{2}) - \sin^2(\frac{\gamma_2}{2})} = \int \frac{-\sin(\frac{\gamma_2}{2}) \, dx}{\cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2})} + \int \frac{\cos(\frac{\gamma_2}{2}) \, dx}{\cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2})} \]

Returning to our list of weapons, change of variable appears first.

\[ u = \cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2}); \quad du = \frac{1}{2} \left[ -\sin(\frac{\gamma_2}{2}) + \cos(\frac{\gamma_2}{2}) \right] \, dx \]

\[ v = \cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2}); \quad dv = \frac{1}{2} \left[ -\sin(\frac{\gamma_2}{2}) - \cos(\frac{\gamma_2}{2}) \right] \, dx \]

The choice is to head toward the form below and to hope that the \textit{remainder} can be vanquished.

\[ \int \frac{dx}{\cos^2(\frac{\gamma_2}{2}) - \sin^2(\frac{\gamma_2}{2})} = \int \frac{-\sin(\frac{\gamma_2}{2}) \\ dx}{\cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2})} + \int \frac{\cos(\frac{\gamma_2}{2}) \\ dx}{\cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2})} \Rightarrow \int \frac{du}{u} - \int \frac{dv}{v} + \text{remainder} \]

\[ = \int \frac{-\sin(\frac{\gamma_2}{2}) + \cos(\frac{\gamma_2}{2}) \\ dx}{\cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2})} - \int \frac{-\sin(\frac{\gamma_2}{2}) - \cos(\frac{\gamma_2}{2}) \\ dx}{\cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2})} \]

\[ = \int \frac{-\sin(\frac{\gamma_2}{2}) - \cos(\frac{\gamma_2}{2}) \\ dx}{\cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2})} + \int \frac{\cos(\frac{\gamma_2}{2}) \\ dx}{\cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2})} \]

The remainder is just the negative of the left-hand side. We have a winner!

\[ 2 \int \frac{dx}{\cos^2(\frac{\gamma_2}{2}) - \sin^2(\frac{\gamma_2}{2})} = \int \frac{[-\sin(\frac{\gamma_2}{2}) + \cos(\frac{\gamma_2}{2})] \, dx}{\cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2})} - \int \frac{[-\sin(\frac{\gamma_2}{2}) - \cos(\frac{\gamma_2}{2})] \, dx}{\cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2})} = 2 \left( \int \frac{du}{u} - \int \frac{dv}{v} \right) \]

\[ \int \frac{dx}{\cos^2(\frac{\gamma_2}{2}) - \sin^2(\frac{\gamma_2}{2})} = \int \frac{dx}{\cos(x)} = \int \frac{du}{u} - \int \frac{dv}{v} = \ln \left[ \cos(\frac{\gamma_2}{2}) + \sin(\frac{\gamma_2}{2}) \right] - \ln \left[ \cos(\frac{\gamma_2}{2}) - \sin(\frac{\gamma_2}{2}) \right] \]
\[
\int \frac{dx}{\cos(x)} = \ln \left[ \cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right) \right] - \ln \left[ \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) \right]
\]

Equivalently: \[
\int \sec(x) \, dx = \ln \left[ \frac{1 + \tan\left(\frac{x}{2}\right)}{1 - \tan\left(\frac{x}{2}\right)} \right]
\]

Patience pays off! It remains unfortunate that no new integrals have been evaluated since the widespread introduction of television.

**Exercise:** Choose \( u = \sec(x) + \tan(x) \). Compute \( du \). Use these results to compute \( \int \sec(x) \, dx \).

Show that this result agrees with the result above this exercise.

**Sample Calculation IB.12: Powers of \( \sin \theta \) and of \( \cos \theta \):**

A very simple, but common integral is of the form: \[
\int_{\theta_1}^{\theta_2} [\sin \theta]^m [\cos \theta]^n \, d\theta.
\]

If \( m \) is odd, than the answer is simple!

\[
\int_{\theta_1}^{\theta_2} [\sin \theta]^m [\cos \theta]^n \, d\theta \rightarrow \int_{\theta_1}^{\theta_2} [\sin \theta]^{(m-1)/2} [\cos \theta]^n \sin \theta \, d\theta = \int_{\theta_1}^{\theta_2} [1 - \cos^2 \theta]^{(m-1)/2} [\cos \theta]^n \sin \theta \, d\theta
\]

As \((m-1)/2\) is an integer, the entire problem is reduced to sums of terms \( \int_{\theta_1}^{\theta_2} [\cos \theta]^n \sin \theta \, d\theta \). Choose \( u = \cos \theta \) and hence \( du = -\sin \theta \, d\theta \). Each term becomes \( \int_{\cos(\theta_1)}^{\cos(\theta_2)} u^n \, du \).

If \( m \) is even, the form of interest becomes \( \int_{\theta_1}^{\theta_2} [1 - \cos^2 \theta]^{(m-1)/2} [\cos \theta]^n \, d\theta \) leading to \( \int_{\theta_1}^{\theta_2} [\cos \theta]^k \, d\theta \). If \( k \) is odd, it is back to the same game, \( \int_{\theta_1}^{\theta_2} [1 - \sin^2 \theta]^{(k-1)/2} \cos \theta \, d\theta \). Here, \( u = \sin \theta \) and \( du = \cos \theta \, d\theta \).

\[
\int_{\sin(\theta_1)}^{\sin(\theta_2)} [1 - u^2]^{(k-1)/2} \, du \text{ which is easy as } (k-1)/2 \text{ is an integer. If } k \text{ is even, you use } \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)
\]

to get to the forms \( \int_{\theta_1}^{\theta_2} [\cos(2\theta)]^l \, d\theta \). You keep running around this loop until the solution is reached.

**Example:** \[
\int_{0}^{\pi} [\sin \theta]^2 [\frac{1}{2}\cos \theta - \frac{1}{2}] \sin \theta \, d\theta \rightarrow \int_{0}^{\pi} [1 - \cos^2 \theta] [\frac{1}{2}\cos^2 \theta - \frac{1}{2}] \sin \theta \, d\theta
\]
\[
\int_{-1}^{1} [1-u^2] (3/2 u^2 - 1/2) \, du = \int_{-1}^{1} [-3/2 u^4 + 2u^2 - 1/2] \, du = -\frac{3}{2} (3/2) + 2(2/2) - 1/2(2) = \frac{-9 + 20 - 15}{15} = \frac{-4}{15}
\]
\[
\int_{0}^{\pi} \sin^2 \theta \left( \frac{3}{2} \cos \theta - \frac{1}{2} \right) \sin \theta \, d\theta \to \frac{-4}{15}
\]

**Converting Sums to Integrals**

It is said that an integral is a sum of little pieces, but some precision is required before the statement becomes useful. Beginning with a function \( f(t) \) and a sequence of values for \( t = \{t_1, t_2, t_3, \ldots, t_N\} \), the sum \( \sum_{i=1}^{N} f(t_i) \) does not represent the integral \( \int_{t_<}^{t_>} f(t) \, dt \) even if a great many closely spaced values of \( t \) are used. Nothing has been included in the sum to represent \( dt \). One requires \( \sum_{i=1}^{i=N} f(t_i) \Delta t_i \) where \( \Delta t_i = \left( \frac{1}{2} \right) [t_{i+1} - t_{i-1}] \) is the average interval between sequential values of \( t \) at \( t_i \). For well-behaved cases, the expression \( \sum_{i=1}^{i=N} f(t_i) \Delta t_i \) approaches the Riemann sum definition of an integral as the \( t \)-axis is chopped up more and more finely. As illustrated below, in the limit that \( \Delta t \) goes to zero, the sum \( \sum_{i=1}^{N} f(t_i) \Delta t_i \) approaches the area under the curve between \( t_< \) and \( t_> \). That is; it represents \( \int_{t_<}^{t_>} f(t) \, dt \) provided the sequence of sums converges, and life is good. The theory of integration is not the topic of this passage. The goal is simply to remind you that the \( \Delta t \) must be factored out of each term that is being summed in order to identify the integrand.
For the discussion of the inner product in the Fourier Series handout, the function \( h(t) = g(t)f(t) \) was considered at \( N \) equally spaced points between \(-T/2\) and \(+T/2\). This form is motivated by the form of the inner product of two 3D vectors expressed in terms of their components:

\[
\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = \sum_{i=1}^{3} A_i B_i
\]

which suggests that the sum of the products of corresponding values of the functions \( f(t) \) and \( g(t) \) \( \rightarrow \) \( \sum_{m=1}^{N} g(t_m) f(t_m) \). This leads to the sum \( \sum_{m=1}^{N} h(t_m) \) where the points \( t_m = -\left(\frac{T}{2}\right) - \left(\frac{T}{2N}\right) + m\left(\frac{T}{N}\right) \) have equal spacing \( \Delta t = \frac{T}{N} \). As the number of terms gets large, the sum must be divided by \( N \) to ensure that the result remains finite. This leads to

\[
\sum_{m=1}^{N} h(t_m) \left(\frac{1}{N}\right).
\]

The rule for converting sums to integrals requires that \( \Delta t = \frac{T}{N} \) be explicitly factored from each term in the sum.

\[
\sum_{m=1}^{N} h(t_m) \left(\frac{1}{N}\right) \rightarrow \sum_{m=1}^{N} \left[ \left(\frac{1}{T}\right) h(t_m) \right] \left(\frac{T}{N}\right) = \sum_{m=1}^{N} \left[ \left(\frac{1}{T}\right) h(t_m) \right] \Delta t
\]

which becomes

\[
\int_{-T/2}^{+T/2} \left[ \left(\frac{1}{T}\right) h(t) \right] dt = \left(\frac{1}{T}\right) \int_{-T/2}^{+T/2} \left[ g(t) f(t) \right] dt \text{ as } N \text{ gets large and } \Delta t \text{ small.}
\]

**Riemann Sum Examples:** (Figures by N. Frigo.) \( \int_{0.3}^{2.4} [x^3 - 4.3 x^2 + 5.4 x + 0.2] \, dx = 4.24568 \)
Change of Variable (substitution) How to guess $u$ … The Golden Guess

A guide to a possibly beneficial choice of a new integration variable is the argument of the most troublesome part of the integrand. Consider:

$$\int \frac{\sin^2(x)}{\sqrt{1 - \cos(x)}} \, dx$$

The $\sin^2(x)$ is, by itself, not too frightening so we choose $u = 1 - \cos(x)$ as it is the argument of the somewhat troubling square root. It follows that $du = \sin(x) \, dx$. Keep the faith, be patient. See how far you can push with your choice. Try to express the remaining factors in the integrand in terms of your new variable.

$$\int \frac{\sin(x)}{\sqrt{u}} (\sin(x) \, dx) = \int \frac{\sin(x)}{\sqrt{u}} \, du = \int \frac{\sqrt{[1 - \cos(x)][1 + \cos(x)]}}{\sqrt{u}} \, du$$

As $u = 1 - \cos(x)$,
\[
\int \frac{\sqrt{[1-\cos(x)][1+\cos(x)]}}{\sqrt{u}} \, du = \int \sqrt{[1+\cos(x)]} \, du = \int \sqrt{2-u} \, du
\]

We make a new change choice: \( v = 2 - u, \)
\[
\int v^{\frac{1}{2}} (-dv) = -\frac{2}{3} v^{\frac{3}{2}} = -\frac{2}{3} [2-u]^{\frac{3}{2}} = -\frac{2}{3} [1+\cos(x)]^{\frac{3}{2}}
\]
\[
\int \frac{\sin^2(x)}{\sqrt{1-\cos(x)}} \, dx = -\frac{2}{3} [1+\cos(x)]^{\frac{3}{2}} + C
\]

One should include the additive constant \( C \) even though it is often omitted in this handout.

**Surface Integrals – parameterizing and area scaling**

Two parameters are required to define a (2D) surface. A general point on the surface is represented as \( \mathbf{r}(u,v) = x(u,v) \mathbf{i} + y(u,v) \mathbf{j} + z(u,v) \mathbf{k}. \) Defining \( d\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \, du \) and \( d\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} \, dv, \) the parameter area patch \( du \, dz \) maps to the area \( dA = |d\mathbf{r}_u \times d\mathbf{r}_v|. \) This correlation has a geometric interpretation in terms of projections in the case that the surface can be represented as \( \mathbf{r}(x,y) = x \mathbf{i} + y \mathbf{j} + z(ux,y) \mathbf{k}. \)

**Surface Integrals – area scaling for projection:**

A problem may require an integration over a surface \( \sigma: f(x,y,z) = C. \) As a surface integral is inherently a two dimensional, an approach is to compute the integral parameterized by the coordinate system for its projection onto a flat plane. First, we set the stage.

**The normal to the surface \( \sigma: \)** The value of \( f(x,y,z) \) is constant for small displacements in its tangent plane so the normal is in the direction of the gradient of \( f(x,y,z), \) the direction in which \( f(x,y,z) \) changes most rapidly.

\[
\text{normal to the surface } \sigma: \quad \hat{n}_\sigma = \frac{\nabla f}{|\nabla f|}
\]

As a first step, the surface \( \sigma \) is to be projected onto the \( x-y \) plane, and the relation between the area patch in the flat plane \( dA = dx \, dy \) and the area \( d\sigma \) in \( f(x,y,z) = C \) that projects onto \( dA \) is developed.

\[
\text{normal to the } x-y \text{ plane } \quad \hat{n}_A = \hat{k}
\]
The process begins by determining the line segments $d\vec{r}_\mu$ and $d\vec{r}_v$ in the surface that project onto $dx\hat{i}$ and $dy\hat{j}$. As the surface is defined by $f(x,y,z) = C$, a small displacement $dx\hat{i} + dy\hat{j} + dz\hat{k}$ lies in the surface only if $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$. For $d\vec{r}_\mu$, $dx$ is $dx$ and $dy$ is zero.

$$d\vec{r}_\mu = dx\hat{i} + 0\hat{j} - \frac{\partial f}{\partial z} dx\hat{k}$$

Similarly, $d\vec{r}_v = 0\hat{i} + dy\hat{j} - \frac{\partial f}{\partial z} dy\hat{k}$.

The ratio of $d\sigma$ to $dA$ follows as $d\sigma = |d\vec{r}_\mu \times d\vec{r}_v| = \left| \frac{\frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}}{\frac{\partial f}{\partial z}} \right| dx\ dy = \frac{\nabla f}{\left| \nabla f \right|} dA = \frac{\partial f}{\partial z} dA$. The $z$ coordinate plays a special role because the surface element $d\sigma$ was projected onto the patch $dA = dx\ dy$ in the $x$-$y$ plane which has the $z$ direction as its normal direction. The equation can be expressed in vector notation so that the more general relation becomes:

$$d\sigma = \frac{\left| \nabla f \right|}{\left| \hat{n}_A \cdot \nabla f \right|} dA = \frac{1}{\left| \hat{n}_A \cdot \hat{n}_\sigma \right|} dA$$

This result is easily understood by considering the projection of a small flat area element onto a plane.

The area patches are viewed edge on and each has a depth $w$. The area $d\sigma = Lw$ projects onto $dA = (L \cos \theta)w$. One concludes that:

$$d\sigma = \frac{1}{\cos \theta} dA = \frac{1}{\left| \hat{n}_A \cdot \hat{n}_\sigma \right|} dA$$
In the differential limit, the segments \( d\vec{r}_\mu \) and \( d\vec{r}_\nu \) are essentially straight. The segments are two edges of a small parallelogram with area equal to the magnitude of the cross product of the segments.
The surface $\sigma$ floating above the $x$-$y$ plane.

$$d\sigma = \frac{1}{|\mathbf{n}_A \cdot \mathbf{n}_\sigma|} dA$$

The surface $A$ with the planar $x$-$y$ coordinate grid.

The patch marked $d\sigma$ projects onto the patch marked $dA$.

**Sample Calculation:** Compute the area of a hemispherical dome over a circle of radius $R$. Polar coordinates are adopted for circle in the plane so $dA = r \, d\phi \, dr$. The hemispherical surface is described by the equation $x^2 + y^2 + z^2 = R^2$ which has as its normal $\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$. The normal to the projection plane is the $\mathbf{k}$ so $\cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$. The conclusion is that:

$$d\sigma = \left( \frac{\sqrt{x^2 + y^2 + z^2}}{z} \right) dA \rightarrow d\sigma = \left( \frac{R}{\sqrt{R^2 - r^2}} \right) dA$$

$$\int d\sigma = \int_0^R \int_0^{2\pi} \left( \frac{R}{\sqrt{R^2 - r^2}} \right) r \, d\phi \, dr = 2\pi \int_0^R \left( \frac{R}{\sqrt{R^2 - r^2}} \right) r \, dr$$
\[ \int d\sigma = 2\pi R \int_{r^2}^{R^2} u^{-\frac{3}{2}} (-\frac{1}{2} du) = 2\pi R u^\frac{1}{2} \bigg|_0^{R^2} = 2\pi R^2 \]

The change of variable was \( u = R^2 - r^2 \) leading to \( du = -2r \, dr \). We found that the area of a hemispherical cap is half the area of a sphere of the same radius, not exactly news.

The inverse cosine factor is one at the pole and it diverges as \( x^2 + y^2 \to R^2 \).

The standard approach for parameterized surface integrals is work directly with \( d\vec{r}_u \) and \( d\vec{r}_v \).

Start by parameterizing using surface in terms of \( u \) and \( v \) and use
\( \vec{r}(u,v) = x(u,v) \hat{i} + y(u,v) \hat{j} + z(u,v) \hat{k} \) to form \( d\vec{r}_u = \frac{\partial \vec{r}}{\partial u} du \) and \( d\vec{r}_v = \frac{\partial \vec{r}}{\partial v} dv \). It follows that \( d\vec{r}_u \times d\vec{r}_v = dA \hat{n} \).

**Tools for the Method of Partial Fractions:**

**Sample Calc1:**

\[
\frac{x^2 + 4x - 2}{x(x+5)(x-2)} = A + \frac{B}{x+5} + \frac{C}{x-2} = \frac{A(x+5)(x-2) + Bx(x-2) + Cx(x+5)}{x(x+5)(x-2)}.
\]

It follows that: \( x^2 + 4x - 2 = A(x+5)(x-2) + Bx(x-2) + Cx(x+5) \). The most direct method to evaluate the constants is to match the coefficients powers of \( x \).

\[
x^2: \quad 1 = A + B + C \quad \text{x: } 4 = 3A + (-2)B + 5C \quad \text{x}^0: \quad -2 = -10A
\]

Clearly, these equations could be solved to identify the constants.

The coefficients of any inverse first power factor can be identified more directly. As \( x \to -5 \), the term with the coefficient \( B \) dominates.

\[
A = \text{Lim}_{x \to -5} \left( \frac{x^2 + 4x - 2}{x(x+5)(x-2)} \right) = \frac{1}{5} \quad \text{B} = \text{Lim}_{x \to -5} \left( \frac{x^2 + 4x - 2}{x(x-2)} \right) = \frac{3}{35} \quad \text{C} = \text{Lim}_{x \to -5} \left( \frac{x^2 + 4x - 2}{x(x+5)} \right) = \frac{5}{7}
\]

**Exercise:** Verify that the coefficients listed above are the correct:

\[
\frac{x^2 + 4x - 2}{x(x+5)(x-2)} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{x-2}
\]

**Sample Calc 2:** This limit method is less effective with higher order factors in the denominator.

Consider:

\[
\frac{x^2 + 4x - 2}{(x+5)(x-2)^2} = \frac{A}{x+5} + \frac{Bx+C}{x-2}
\]

The \( A \) coefficient can be found as before:

\[
A = \text{Lim}_{x \to -5} \left( \frac{x^2 + 4x - 2}{(x-2)^2} \right) = \frac{3}{49}
\]

Alternately, the target equation must be valid for all \( x \).
\[ x^2 + 4x - 2 = A(x - 2)(x - 2) + (Bx + C)(x + 5) \]

First set \( x = -5 \) to make the last factor zero.

\[ (-5)^2 + 4(-5) - 2 = A(-5 - 2)(-5 - 2) + (B(-5) + C)(0) \Rightarrow A = \frac{3}{49} \]

This step reproduced a prior result. Next we set \( x = 0 \) to simplify \( Bx + C \).

\[ x^2 + 4x - 2 = A(x - 2)(x - 2) + (Bx + C)(x + 5) \Rightarrow -2 = A(-2)(-2) + 5C \]
\[ -2 - \frac{12}{49} = 5C \Rightarrow C = -\frac{22}{49} \]

The final parameter \( B \) can be found by choosing yet another value of \( x \); say: \( x = 1 \).

\[ x^2 + 4x - 2 = A(x - 2)(x - 2) + (Bx + C)(x + 5) \]
\[ 1 + 4 - 2 = A + (B + C)(1 + 5) \Rightarrow B = \left(\frac{1}{6}\right)(3 - A) - C \]
\[ B = \left(\frac{1}{6}\right)\left(3 - \left[\frac{3}{49}\right]\right) - \left(-\frac{22}{49}\right) = \frac{46}{49} \]

**Exercise:** Give the equations that represent matching the coefficients of each power of \( x \) in:
\[ x^2 + 4x - 2 = A(x - 2)(x - 2) + (Bx + C)(x + 5) \]. Verify that \( A = \frac{3}{49}; B = \frac{-22}{49}; \) and \( C = \frac{46}{49} \) satisfy those relations.  *NOT CHECKED !!!!*

**Advanced Method of Partial Fractions:**

This section is just a placeholder for a promised development. Nonetheless, it does include material that suggests how one might proceed.

**SECTION NOT READY; DO NOT READ!**

Integrands with complicated denominators present special problems. Partial fraction is effective for integrands that are rational functions \( r(x) \), the ratio of an \( m \)th order polynomial numerator \( u_m(x) \) to a \( n \)th order polynomial denominator \( v_n(x) \). The method may even apply when the polynomial in the numerator is multiplied by a more general function such as \( \sin(x) \). The denominator should not be more complicated than the \( n \)th order polynomial form \( v_n(x) \). For the procedure to work, the denominator should be a polynomial of order higher than that of the polynomial in the numerator. This limitation is not a serious one however. If \( m > n \), divide the denominator into the numerator.
\[
\frac{u_k(x)}{v_n(x)} = w_{k-n}(x) + R_{n-1}(x)
\]

The result is a polynomial plus a remainder term which is a polynomial of order \( n - 1 \). The \( w_{k-n}(x) \) term is easily integrated. The remainder term becomes a \( u'(m)(x) \) where \( m \) is less than the order of the denominator. The procedure can now be applied to the new \( u'(m)/ v_n(x) \) term.

It is assumed that \( v_n(x) \) factors in the form,

\[
v_n(x) = (x-a_1)^{n_1} (x-a_2)^{n_2} \ldots (x-a_N)^{n_N} (x+b_1)^2 \ldots (x+b_M)^2.
\]

It follows that: \( n = \sum_{i=1}^{N} n_i + 2M \). Solve the equations necessary to place the integrand in the form:

\[
\frac{u_k(x)}{v_n(x)} = \left( \frac{d_1(x)}{(x-a_1)^{n_1}} \right) + \left( \frac{d_2(x)}{(x-a_2)^{n_2}} \right) + \ldots + \left( \frac{d_N(x)}{(x-a_N)^{n_N}} \right) + \left( \frac{B_i x + C_i}{(x+b_i)^2} \right) + \ldots + \left( \frac{B_M x + C_M}{(x+b_M)^2} \right)
\]

where the \( d_i(x) \) are polynomials of order \( n_i - 1 \). A forest of coefficients must be assigned labels and determined by extensive algebraic acrobatics. It is painful, but worth the effort if no other path to the goal (computing the integral) is available.

Equivalently, each factor \( \left( \frac{d_i(x)}{(x-a_i)^{n_i}} \right) \) can be replaced by \( \left( \frac{d_{i_1}}{(x-a_i)^{n_i}} \right) \left( \frac{d_{i_2}}{(x-a_i)^{n_i}} \right) \ldots \left( \frac{d_{i_m}}{(x-a_i)^{n_i}} \right) \) where the \( d_{im} \) are scalar constants.

A special, simpler case as an example:

\[
v_n(x) = (x-a_1)^{n_1} (x-a_2)^{n_2} \ldots (x-a_N)^{n_N}
\]

\[
\frac{u_k(x)}{v_n(x)} = \left( \frac{d_1(x)}{(x-a_1)^{n_1}} \right) + \left( \frac{d_2(x)}{(x-a_2)^{n_2}} \right) + \ldots + \left( \frac{d_N(x)}{(x-a_N)^{n_N}} \right)
\]

\[
u_k(x) = \left( \frac{d_1(x)}{(x-a_1)^{n_1}} \right) + \left( \frac{d_2(x)}{(x-a_2)^{n_1}} \right) + \ldots + \left( \frac{d_N(x)}{(x-a_N)^{n_1}} \right) \prod_{i=1}^{N} (x-a_j)^{n_j}
\]

\[
u_k(x) = d_1(x) \prod_{i=1}^{N} (x-a_j)^{n_j} + d_2(x) \prod_{i=2}^{N} (x-a_j)^{n_j} + \ldots + d_N(x) \prod_{i=1}^{N} (x-a_j)^{n_j}
\]

For each case in which \( n_j = 1 \), set \( x = a_j \). All the terms vanish except for the one for \( d_j(x) \).
In this case, \( d_j \) is just the constant: 
\[
d_j = u_k(a_j) \prod_{i \neq j} (a_j - a_i)^n.
\]

**Liebniz Rule:** Consider an integral with an integrand and limits that depend on a parameter \( t \). The Liebniz rule provides the derivative on the integral with respect to that parameter. Given

\[
I(t) = \int_{g(t)}^{h(t)} f(x,t) \, dx
\]

the rule yields \( \frac{\partial I}{\partial t} \). Below, we plot \( I(t) \) and \( I(t + \Delta t) \). The band between the two integrands is \( \frac{\partial f}{\partial t} \Delta t \). The lower and upper limits increase by \( \frac{\partial g}{\partial t} \Delta t \) and \( \frac{\partial h}{\partial t} \Delta t \) where the integrand has values \( f(g(t), t) \) and \( f(h(t)) \) respectively.

When \( t \) increases by \( \Delta t \) the area under the curve increases by the top band on area and the right end band and decreases by the left end band.

\[
I(t + \Delta t) - I(t) = \int_{g(t+\Delta t)}^{h(t+\Delta t)} f(x, t + \Delta t) \, dx - \int_{g(t)}^{h(t)} f(x, t) \, dx - \int_{g(t)}^{h(t)} \left( \frac{\partial f}{\partial t} \Delta t \right) \, dx + f(h(t), t) \frac{\partial h}{\partial t} \Delta t - f(g(t), t) \frac{\partial g}{\partial t} \Delta t
\]
Note that the small patches at that lie in the top band and either the left or right end band have areas of order \((\Delta t)^2\).

It follows that:

\[
\frac{I(t + \Delta t) - I(t)}{\Delta t} = \int_{g(t)}^{h(t)} \frac{\partial f}{\partial t} \, dx + f(h(t), t) \frac{\partial h}{\partial t} - f(g(t), t) \frac{\partial g}{\partial t} + \frac{O(\Delta t^2)}{\Delta t}
\]

The final term vanishes in the limit \(\Delta t \to 0\), we find Liebniz rule.

\[
\frac{\partial I(t)}{\partial t} = \int_{g(t)}^{h(t)} \frac{\partial f}{\partial t} \, dx + f(h(t), t) \frac{\partial h}{\partial t} - f(g(t), t) \frac{\partial g}{\partial t}
\]
Tabular Method for Integration by Parts


Example 1: \( \int x^2 \cos(x) \, dx \)

<table>
<thead>
<tr>
<th>( D )</th>
<th>( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( 2x )</td>
<td>( -\sin x )</td>
</tr>
<tr>
<td>( 2 )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( -\sin x )</td>
</tr>
</tbody>
</table>

\[
\int x^2 \cos x \, dx = x^2 \sin x - 2x(-\cos x) + 2(-\sin x) + C \\
= x^2 \sin x + 2x \cos x - 2\sin x + C
\]

Example 2:

<table>
<thead>
<tr>
<th>( D )</th>
<th>( I )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^3 + 2x )</td>
<td>( e^{2x} )</td>
</tr>
<tr>
<td>( 3x^2 + 2 )</td>
<td>( e^{2x}/2 )</td>
</tr>
<tr>
<td>( 6x )</td>
<td>( e^{2x}/4 )</td>
</tr>
<tr>
<td>( 6 )</td>
<td>( e^{2x}/8 )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( e^{2x}/16 )</td>
</tr>
</tbody>
</table>

\[
\int (x^3 + 2x) \, e^{2x} \, dx = \frac{x^3 + 2x}{2} \, e^{2x} - \frac{(3x^2 + 2)}{4} \, e^{2x} + \frac{6x}{8} \, e^{2x} - \frac{6}{16} \, e^{2x} + C \\
= \frac{e^{2x}}{8} (4x^3 - 6x^2 + 14x - 7) + C
\]

https://www.youtube.com/watch?v=L2_JCyMfMzA
### The General Method: \( \int f(x)g(x)\,dx \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Derivative(^1)</th>
<th>Integrated(^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( f(x) )</td>
<td>( X \text{ over and down} )</td>
</tr>
<tr>
<td>1</td>
<td>( f^{[1]}(x) )</td>
<td>- (( X \text{ over and down} ))</td>
</tr>
<tr>
<td>2</td>
<td>( f^{[2]}(x) )</td>
<td>( X \text{ over and down} )</td>
</tr>
<tr>
<td>3</td>
<td>( f^{[3]}(x) )</td>
<td>- (( X \text{ over and down} ))</td>
</tr>
</tbody>
</table>

\[
\int f(x)g(x)\,dx = f(x)G^{[1]}(x) - f^{[1]}(x)G^{[2]}(x) + f^{[2]}(x)G^{[3]}(x) - f^{[3]}(x)G^{[4]}(x) + f^{[4]}(x)G^{[5]}(x) - \ldots \]

Continue until \( f^{[n]}(x) = 0 \) or the form \( \text{constant} \times \int f(x)g(x)\,dx \) appears. In the later case, solve for \( \left(1 - \text{constant}\right) \times \int f(x)g(x)\,dx \).

\[
\int x^2 \cos(x) \,dx
\]

<table>
<thead>
<tr>
<th>Step</th>
<th>Derivative(^1)</th>
<th>Integrated(^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( x^2 )</td>
<td>( X \text{ one over and one down} )</td>
</tr>
<tr>
<td>1</td>
<td>2 ( x )</td>
<td>- (( X \text{ one over and one down} ))</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>( X \text{ one over and one down} )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>- (( X \text{ one over and one down} ))</td>
</tr>
</tbody>
</table>

\[
\int x^2 \cos(x) \,dx = x^2 \sin(x) - 2x (-\cos(x) + 2 (-\sin(x))) + 0 + C
\]
\[ = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C \quad \ldots . \]

Problems

1.) Compute the integral \( \int e^{-t} \cos(t) \, dt \). Use integration by parts twice. Compute \( \int e^{-kt} \cos(t) \, dt \).

3.) Prepare a sketch similar to Figure IB.1 to represent \( I(t_0) \), the integral of the function \( f(x, t_0) \) from \( a(t_0) \) to \( b(t_0) \). Overlay a representation of the integral \( I(t_0 + \Delta t) \), the integral of the function \( f(x, t_0 + \Delta t) \) from \( a(t_0 + \Delta t) \) to \( b(t_0 + \Delta t) \). (The functions \( f(x, t) \), \( a(t) \) and \( b(t) \) are all assumed to be continuous and differentiable.) Compare to show that

\[
I(t_0 + \Delta t) - I(t_0) \approx \int_{a(t_0)}^{b(t_0)} [f(x, t_0 + \Delta t) - f(x, t_0)] dx + f(x, b)[b(t_0 + \Delta t) - b(t_0)] - f(x, a)[a(t_0 + \Delta t) - a(t_0)]
\]

Consider \( \frac{dI}{dt} = \lim_{\Delta t \to 0} \left[ \frac{I(t_0 + \Delta t) - I(t_0)}{\Delta t} \right] \) and relate your results to Leibniz rule.

4.) Evaluate \( \int_{-\pi/4}^{\pi/4} \sec^2(\theta) \, d\theta \). Try \( u = \sec(\theta) \). Do not abandon the choice easily. \textit{Answer:} 2.

Note that the choice for change of variable follows from our standard first guess, the argument of the most complicated function in the integrand. One might think that is \( \theta \), but that is where we started. That leaves \( (\sec\theta)^2 \). The new variable \( \sec\theta \) is the argument of squared.

6.) Evaluate \( \int \sec(\theta) \, d\theta \). Hint: \( \cos(2\theta) = \cos^2\theta - \sin^2\theta \).
7.) Compute $\int \frac{x\,dx}{x^3-3x-2}$.  

Answer: $\left(\frac{2}{\pi}\right)\ln\left[\frac{2-x}{1+x}\right] - \left(\frac{1}{3}\right)\frac{1}{1+x}$

8.) One can compute the arc length $L$ of a curve in the x-y plane if the curve is specified by:

- $\{x(s), y(s)\}$ for $a \leq s \leq b$  
  
  $$L = \int_a^b \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} \, ds$$

- $y(x)$ for $a \leq x \leq b$  
  
  $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

- $x(y)$ for $a \leq y \leq b$  
  
  $$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \, dy$$

Compute the arc length $L$ along the path $x = \frac{1}{3} y^3 + \frac{1}{4} y^{-1}$ for $1 < y < 3$. Answer: $\frac{53}{6}$.

9.) Compute: 

a.) $\int [\sin(x)]^2 \, dx$  

b.) $\int [\sin(x)]^n \cos(x) \, dx$  

c.) $\int [\sin(x)]^4 [\cos(x)]^3 \, dx$

10.) The following sum is to be approximated by an integral.

$$\sum_{m=10}^{m=40} \frac{1}{m^3} \rightarrow \int_{m_{10}}^{m_{40}} m^{-3} \, dm$$

How many terms are included in the sum? What is the change in the value of $m$ from one term to the next term? Assign limits to the integral and estimate the sum. Prepare a sketch that motivates your choice of integration limits. (The sum is approximately $5.22 \times 10^{-3}$.)

Mathematica: $N[\text{Sum}[m^(-3),\{m,10,40\}]] = 0.00522013$.

11.) The following sum is to be approximated by an integral.

$$\sum_{m=11}^{m=45} \frac{1}{m^3} \rightarrow \int_{m_{11}}^{m_{45}} m^{-3} \left(a \, dm\right)$$

How many terms are included in the sum? What is the change in the value of $m$ from one term to the next term? Assign limits to the integral and estimate the sum. Prepare a sketch that motivates your choice of integration limits.
\[ \text{Sum}\{m^{(-3)}, \{m, 11, 45, 2\}\} = 0.00235747. \]

One term is to be added to the sum each time \( m \) increases in value by two. ( \( a\ dm = 1 \) for \( dm = 2 \).) Using just \( dm \) would ‘add a term’ for each unit change in the value of \( m \).

12.) Use trig substitutions to evaluate \( \int_0^1 \frac{dx}{\sqrt{1-x^2}} \) and \( \int_0^1 \frac{dx}{\sqrt{1-x}} \).

13.) Compute the integral \( \int_{-\infty}^{\infty} \text{sech}(x) \, dx \). Choose \( u = e^{-x} \).

14.) Evaluate \( \int_0^\infty e^{-u} \cos(xu) \, du \). Answer: \( \frac{1}{1+x^2} \)

15.) Use the answer for the previous problem as a basis to find \( \int_0^\infty u e^{-u} \sin(xu) \, du \).

16.) Compute \( \int_0^\infty u e^{-u} \cos(xu) \, du \), \( \int_0^\infty u^2 e^{-u} \cos(xu) \, du \) and \( \int_0^\infty u^3 e^{-u} \cos(xu) \, du \).

17.) Compute \( \int_0^\infty u^{-1} e^{-u} \sin(xu) \, du \). What can be said about \( \int_0^\infty u^{-1} e^{-u} \cos(xu) \, du \)?

18.) \textbf{IB.9: Hyperbolic Function Substitution:} \[ \int_{x_1}^{x_2} \frac{dx}{\sqrt{x^2-a^2}} = \int_{x_1/a}^{x_2/a} \frac{d\left(\frac{\gamma}{a}\right)}{\sqrt{\left(\frac{\gamma}{a}\right)^2-1}} \]

\[ \sqrt{a^2+x^2} \Rightarrow \sinh(u) = \frac{x}{a} \text{ as } 1 + (\sinh(u))^2 = (\cosh(u))^2 \text{ and } d(\sinh(u)) = \cosh(u) \, du \]

\[ \sqrt{a^2-x^2} \Rightarrow \tanh(u) = \frac{x}{a} \text{ as } 1 - (\tanh(u))^2 = (\sinh(u))^2 \text{ and } d(\tanh(u)) = (\sech(u))^2 \, du \]

\[ \sqrt{x^2-a^2} \Rightarrow \cosh(u) = \frac{x}{a} \text{ as } (\cosh(u))^2 - 1 = (\sinh(u))^2 \text{ and } d(\cosh(u)) = \sinh(u) \, du \]

Based on the templates, select a change of variable. The transformed limits become inverse hyperbolic functions of the original limits. Use the methods illustrated in sample calculation IB.9 to develop the form of that inverse function. Requires \(|x| > |a| \) throughout.

\[ \int_{x_1}^{x_2} \frac{dx}{\sqrt{x^2-a^2}} = \ln \left[ \frac{x_2 + \sqrt{x_2^2-a^2}}{x_1 + \sqrt{x_1^2-a^2}} \right] \quad \text{assume } x_1 \text{ and } x_2 > |a| \]
19.) Evaluate \( \int_0^a x^2 \sin^2(kx) \, dx \). Use a mixed method approach beginning with trigonometric identities.

20.) Evaluate: \( \int \ln(x) \, dx \). Change of variable is our first choice. The argument of the most troubling function in the integrand is the standard first guess. Here, all we can choose is \( u = \ln(x) \). What is \( du \)? What is \( x \) expressed as a function of \( u \)? Solve to find \( dx \) as the product of a function of \( u \) and \( du \). Which technique is needed to complete the \( u \) integration? After completing the integration in terms of \( u \), replace \( u \) by \( \ln(x) \) everywhere to transform your result into standard form.

21.) **Simple Change of variable Examples** (due to Feynman):
   a.) \( \int [1 + 2t]^3 \, dt \)
   b.) \( \int \sqrt{1 + 5t} \, dt \)

22.) Compute \( \int \frac{dx}{\sqrt{1-x^2}} \). Use \( \cos^2(u) = 1 - \sin^2(u) \) and choose \( x = \sin(u) \). Hence \( u = \sin^{-1}(x) \). What is \( du \)?

23.) Compute \( \int \frac{dx}{\sqrt{1+x^2}} \). Use \( \cosh^2(u) = 1 + \sinh^2(u) \) and choose \( x = \sinh(u) \). Hence \( u = \sinh^{-1}(x) \). What is \( du \)? Given that \( \sinh(u) = \frac{1}{2} [ e^u - e^{-u} ] = x \), invert the relation to show that \( x = \ln[x + (1 + x^2)^{1/2}] \). Conclude that \( \ln[x + (1 + x^2)^{1/2}] = \sinh^{-1}(x) \).

24.) **Complete the Square plus ‘Trig’ Substitution.** (Study the previous problem.)
   a.) Compute \( \int \frac{dx}{\sqrt{ax^2 + bx + c}} \). Answer: \( \ln \left[ \frac{a x + \sqrt{a} \sqrt{a x^2 + bx + c}}{\sqrt{a}} \right] \)
   b.) Compute \( \int \frac{dx}{\sqrt{a x^2 + bx + c}} \). Answer: \( \text{ArcTan} \left[ \frac{\sqrt{a} x}{\sqrt{b x + c}} \right] \)
25.) More substitution (change of variable).

a.) \( \int (a+b \, x)^n \, dx \)

b.) \( \int \sin(a+b \, x) \, dx \)

c.) \( \int x \, e^{(a+b) \, x} \, dx \)

d.) \( \int x \, e^{(a+b) \, x^2} \, dx \)

e.) \( \int \frac{dx}{1+x^2} \)

f.) \( \int \frac{dx}{a+b \, x^2} \)

26.)

a.) Show that \( \int [\cos(x)]^n \, dx = \frac{1}{n} [\cos(x)]^{n-1} \sin(x) + \frac{n-1}{n} \int [\cos(x)]^{n-2} \, dx + C \). Hint: Integrate by parts using \( du = \cos(x) \, dx \). What is \( v \)?

b.) Use the result to compute \( \int_0^{\pi/2} [\cos(x)]^2 \, dx \).

c.) Use the result to compute \( \int_0^{\pi/2} [\cos(x)]^3 \, dx \).

d.) Compute \( \int_0^{\pi/2} [\cos(x)]^2 \, dx \) using \( \cos^2{x} = \frac{1}{2} [1 + \cos(2x)] \).

e.) Compute \( \int_0^{\pi/2} [\cos(x)]^3 \, dx \) using \( \cos^2{x} = 1 - \sin^2{x} \).

27.) Parameter Calculus: \( (\int_0^\infty e^{-a \, x^2} \, dx = \frac{\sqrt{\pi}}{2a}) \) is a standard integral, and its evaluation is presented in the definite integrals handout. Use the parameter calculus approach to compute \( \int_0^\infty x^2 \, e^{-x^2} \, dx \), \( \int_0^\infty x^4 \, e^{-x^2} \, dx \) and \( \int_0^\infty x^{2n} \, e^{-x^2} \, dx \) where \( n \) is an integer.

\[
\int_0^\infty x^{2n} \, e^{-x^2} \, dx = (2n-1)!! \left( \frac{\sqrt{\pi}}{2^{n+1}} \right) \text{ where } m!! = m \cdot (m-2) \cdot (m-4) \cdot \ldots \cdot (\text{ending with 2 or 1}).
\]

28.) Parameter Calculus: \( I(a) = \int \frac{du}{(a-u^2)} \) is a standard integral, and it can be evaluated using the method of partial fractions. \( I(a) = \frac{1}{2\sqrt{a}} \int \left( \frac{1}{(\sqrt{a}+u)} + \frac{1}{(\sqrt{a}-u)} \right) \, du \).

a.) Evaluate \( I(a) \).

b.) Compute \( \frac{dI(a)}{da} \) and \( \frac{d^2 I(a)}{da^2} \).

c.) Compute \( \frac{d}{da} \left( \int \frac{du}{(a-u^2)} \right) \) and \( \frac{d^2}{da^2} \left( \int \frac{du}{(a-u^2)} \right) \).

d.) Evaluate each result for \( a = 1 \).
29.) Parameter Calculus: \( J(a) = \int \frac{du}{(a^2-u^2)} \) is a standard integral, and it can be evaluated using the method of partial fractions. \( J(a) = \frac{1}{2a} \int \left( \frac{1}{(a+u)} + \frac{1}{(a-u)} \right) du \).

a.) Evaluate \( J(a) \).

b.) Compute \( \frac{dJ(a)}{da} \) and \( \frac{d^2 J(a)}{da^2} \).

c.) Compute \( \frac{d}{da} \int \frac{du}{(a^2-u^2)} \) and \( \frac{d^2}{da^2} \int \frac{du}{(a^2-u^2)} \).

d.) Evaluate each result for \( a = 1 \).

30.) Using Identities: \( J = \int \frac{du}{(1^2-u^2)} \) is a standard integral, and it can be evaluated using the method of partial fractions. \( J = \frac{1}{2} \int \left( \frac{1}{(1+u)} + \frac{1}{(1-u)} \right) du \rightarrow \frac{1}{2} \ln \left( \frac{(1+u)}{(1-u)} \right) \).

a.) Suppose that \( u = \tanh(x) \). That is: \( u = \frac{e^x-e^{-x}}{e^x+e^{-x}} \). Solve the expression for \( x(u) \)

b.) Express \( J \) in terms of the \( \text{arctanh}(u) \).

31.) Considering the Schwarzschild metric in general relativity, the following integral arises when one computes the radial distance for a given change in radial coordinate: \( \int \frac{dr}{\sqrt{1-\frac{2\mu}{r}}} \). Change of variable is one of our most powerful methods. a.) Choose \( x = \sqrt{1-\frac{2\mu}{r}} \) and compute \( dx \).

b.) Compute the radial distance between \( r = 2\mu \) and \( r = 3\mu \) in the Schwarzschild metric.

\( \ell_r = \int_{2\mu}^{3\mu} \frac{dr}{\sqrt{1-\frac{2\mu}{r}}} \). Hint: Study the two problems above this one.

Answer (not checked!): \( \ell_r = \mu \left[ \sqrt{3} + \log \left( 2 + \sqrt{3} \right) \right] \)

32.) Parameter Calculus: \( \int t \sin(mt) \, dt = -\frac{d}{dm} \int \cos(mt) \, dt \). Extend this to find expressions for:

\( \int t^k \sin(mt) \, dt \) for \( k \) even and for \( k \) odd.
\[ \int t^k \cos(mt) \, dt \text{ for } k \text{ even and for } k \text{ odd.} \]

33.) a.) Integrate \( \int x^n \sin(kx) \, dx \) by parts once to yield an integral with a lower power of \( x \).

b.) Integrate \( \int x^n \cos(kx) \, dx \) by parts once to yield an integral with a lower power of \( x \).

c.) Use your results iteratively to compute \( \int x^4 \sin(kx) \, dx \) and \( \int x^3 \sin(kx) \, dx \).

34.) Persevere. It is often the case that one must try and try again to evaluate an integral. I will guide you through an example. A.) Verify the evaluation below.

\[ g = \int \frac{dx}{x \sqrt{9x^2 + 7}} \rightarrow \frac{\ln[3x] - \ln[\sqrt{7} + \sqrt{9x^2 + 7}]}{\sqrt{7}} + C \quad \text{not checked!} \]

Step 1: Change variable to the argument of … . \( u = 9x^2 + 7 \).

Step 2: Change again. \( w = \sqrt{u} \)

Step 3: Partial fractions. Use the properties of \( \ln[x] \).

b.) Show that this form is equivalent: \( g = \int \frac{dx}{x \sqrt{9x^2 + 7}} \rightarrow \frac{\ln[x] - \ln[\sqrt{7} \sqrt{9x^2 + 7}]}{\sqrt{7}} + C' \)

Note that we should use absolute values of the arguments of natural log.

35.) Persevere Two. It is often the case that one must try and try again to evaluate an integral. I will guide you through an example. A.) Evaluate:

\[ g = \int \frac{dx}{x \sqrt{9x^2 - 7}} \rightarrow \frac{\ln[x] - \ln[\sqrt{7} \sqrt{9x^2 - 7}]}{\sqrt{7}} + C \quad \text{not checked!} \]

Step 1: Change variable to the argument of … . \( u = 9x^2 - 7 \).

Step 2: Change again. \( w = \sqrt{u} \)

Step 3: Derivative of arctangent.

b.) Compare with the answer to the previous problem. Show that \( \tan[i \, x] = -\tanh[x] \).

\( \tanh[x] = \frac{e^x - e^{-x}}{e^x + e^{-x}} = u \). Solve for \( x(u) \) to display the functional form of \( \tanh^{-1}[u] \).

\( \Rightarrow \) There is a kinship between tangent and logarithm.

36.) Consider the function \( f(x) = 12 - 2 \, x^2 \) and compute the Riemann lower sum for the integral:
\[ \int_0^2 f(x) \, dx \] when the interval is divided into \( n \) equal width strips. Take the limit \( n \to \infty \). Use the results: 
\[ \sum_{k=1}^{n} k^0 = \sum_{k=1}^{n} 1 = n; \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}; \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(n+2)}{6}. \]

37.) Assume the forms: 
\[ \sum_{k=1}^{n} k^0 = a_0 n; \quad \sum_{k=1}^{n} k = a_1 n + b_1 n^2; \quad \sum_{k=1}^{n} k^2 = a_2 n + b_2 n^2 + c_2 n^3. \]
Evaluate the constants in each case. For example, evaluate the expression for \( \Sigma k^2 \) for \( n = 1, 2 \) and \( 3 \). The three equations can be solved to yield \( \{a_2, b_2, c_2\} \).

38.) a.) Evaluate: 
\[ \int_1^2 \frac{dx}{x \sqrt{x^2 - a^2}} \]
b.) Evaluate: 
\[ \int_1^2 \frac{dx}{x \sqrt{x^2 + a^2}} \]

39.) Begin with \( \sec^{-1}(\sec(x/a)) = x/a \). Compute the derivative of \( \sec^{-1}(u) \). Use your results to deduce a more friendly expression for 
\[ \int_1^2 \frac{dx}{x \sqrt{x^2 - a^2}} . \]

40.) **Riemann Sum:** Show that the upper Riemann sum for \( \int_1^3 e^x \, dx \) using \( n \) equal width vertical strips can be represented as 
\[ \sum_{k=1}^{n} e^{1+k(\gamma_n)} (\gamma_n). \]

a.) Show that the coordinate of the right edge of the \( k \)th vertical strip is \( 1 + k \left( \frac{2}{n} \right) \). 
b.) Show that 
\[ \sum_{k=0}^{n-1} e^{1+k(\gamma_n)} = \frac{1-e^{n(\gamma_n)}}{1-e^{(\gamma_n)}}. \]
c.) Show that \( \lim_{n \to \infty} \left( \frac{(\gamma_n)}{n} e^{\gamma_n} \right) = -1 \).

You may use L’Hospital’s Rule. d.) Complete the evaluation of 
\[ \sum_{k=1}^{n} e^{1+k(\gamma_n)} (\gamma_n) \] in the limit \( n \to \infty \).

41.) Find the total area between the \( x \) axis and the curve \( y = \sqrt{x - 3} \) over the range \( 0 \leq x \leq 16 \).

Total: \( 12 \frac{2}{3} \).
42.) Evaluate the integral \( \int_{1}^{9} \frac{t^2 + 1}{\sqrt{t}} \, dt \) directly and by adopting the change of variable \( u = \sqrt{t} \). (14.4)

43.) Consider the function \( 2x^2 + x - 6 \) and \( x^2 - 4 \).
   a.) Find the values of \( x \) for which these two curves intersect.
   b.) Compute the area between the two curves for \( x \) between the two intersections.

9½

44.) Integration by Parts: One often needs to integrate by parts several times in order to reach the final answer. In these cases, one must take care to continue to integrate by parts in the same sense so that one does not just undo an earlier integration in a later step. Another strategy is demonstrated by the integration \( \int x \sin^{-1}(x) \, dx \). A.) In the first step one must choose \( du = x \) and \( v = \sin^{-1}(x) \). Why?
   B.) In the next step one needs to reshuffle factors between the new \( u \) and \( v \). Do so and complete the integration.

45.) Evaluate \( \int \csc(x) \, dx \). Use \( \sin(x) = 2 \sin(\frac{1}{2}x) \cos(\frac{1}{2}x) \) and partial fractions.

46.) Evaluate \( \int \sec(x) \, dx \). Use \( u = \sec(x) + \tan(x) \); compute \( du \). How is \( du \) related to \( u \)?

47.) a.) Evaluate \( \int \sec^3(x) \, dx \). Use \( u = \sec(x) \) and \( dv = \sec^2(x) \). Afterward, use a trig identity for \( \sec^2(x) \). Study your equation and the previous problem. b.) Use \( u = \tan\theta \) to transform the integral to the form \( \int \frac{1}{\sqrt{1 + u^2}} \, du \). Evaluate the integral.

48.) Evaluate \( \int \frac{x}{x^3 + 1} \, dx \). Use integration by partial fractions. There will be a first order and a second order denominator. The second order part can be attacked using change of variable, but there will be
some change left over. The leftover can be attacked be completing the square. The result may be related to inverse tangent.

49.) Consider the region of the $x$-$y$ plane bounded by $x = 0; y = 2$ and $y = 2 \sin(x)$. Compute the volume of revolution when this region is rotated about the $y$ axis. a.) Show that an expression for that volume is \[ \int_0^{\pi/2} \pi x^2 \frac{dy}{dx} \, dx \] where $y = 2 \sin(x)$. c.) Compute the volume directly as \[ \int_0^2 \pi \left[ \sin^{-1} \left( \frac{y}{2} \right) \right]^2 \, dy \]. Clearly the change of variable required is $w = \sin^{-1} u$ (after the trivial change $u = \frac{y}{2}$). Multiply and divide by each factor that needs to be present. d.) Compare the integrals in each case.

50.) Compute each of the integrals.

a.) \[ \int \frac{\sin^2(x)}{\sqrt{1-\cos(x)}} \, dx \]

b.) \[ \int_{\pi/2}^{\pi} \frac{\sin^2(x)}{\sqrt{1-\cos(x)}} \, dx \]

c.) \[ \int_{\pi}^{3\pi/2} \frac{\sin^2(x)}{\sqrt{1-\cos(x)}} \, dx \] (not checked)

**convert to sin(x/2) and cos(x/2)**

51.) a.) Show that \[ \frac{d\sin^{-1}(x)}{dx} = \frac{1}{\sqrt{1-x^2}} \]. b.) Compute \[ \int \frac{dx}{\sqrt{1-x^2}} \] using the change of variable $x = \sin(u)$.

c.) Show that \[ \frac{d\sec^{-1}(x)}{dx} = \frac{1}{x\sqrt{x^2-1}} \]. d.) Compute \[ \int \frac{dx}{x\sqrt{x^2-1}} \] using the change of variable $x = \sec(u)$.

51.) **Parameter Calculus:** A.) Compute \[ \int_0^\infty e^{-st} \cos(\omega t) \, dt \] as \[ \frac{1}{2} \int_0^\infty e^{-st} \left[ e^{i\omega t} - e^{-i\omega t} \right] \, dt \]. Given that \[ \int_0^\infty e^{-st} \cos(\omega t) \, dt = \frac{S}{s^2 + \omega^2} \]. B.) Compute \[ \frac{\partial}{\partial \omega} \] of each side of this equation. C.) What is \[ \int_0^\infty e^{-st} \left\{ t \sin(\omega t) \right\} \, dt \]? D.) What is \[ \int_0^\infty e^{-st} \left\{ t^2 \cos(\omega t) \right\} \, dt \]?

52.) \[ \int \frac{dx}{1-\sin(x)} \] Recommend that you consider algebraic as well as trig identities. \tan(x) + \sec(x)

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<table>
<thead>
<tr>
<th><strong>Method D1:</strong> Simple integration by parts</th>
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<tbody>
<tr>
<td>[ \int \ln(x) , dx = x \ln(x) - \int x \left( \frac{1}{x} \right) dx = x \ln(x) - x ]</td>
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<tr>
<th><strong>Method D2:</strong> Simple change of variable</th>
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\[
\int \ln(x) \, dx = \int u \left[ \frac{df}{dx} \right]^{-1} \, du \quad \text{with} \quad x = f^{-1}(u); \quad u = f(x), \ du = \frac{df}{dx} \, dx
\]

\[
\int \ln(x) \, dx = \int u \left[ \frac{1}{x} \right]^{-1} \, du = \int u \, x \, du = \int u \, e^u \, du
\]

\[
\int \ln(x) \, dx = u \, e^u - \int e^u \, du = [u \, e^u - e^u]_{u=\ln(x)} = x \ln(x) - x
\]

53.) Verify that \( \int \sin^{-1}(x) \, dx = \sqrt{1-x^2} + x \sin^{-1}(x) + c. \) Derive the relation directly by integrating using the by parts method. \[\int \sin^{-1}(x) \, dx = \int u \, dv, \ u = \sin^{-1}(x), \ dv = dx\]

54.) Evaluate \( \int \tan^{-1}(x) \, dx \) \[x \tan^{-1}(x) - \frac{1}{2} \ln|1-x^2|\]

55.) Evaluate \( \int \ln(x) \, dx \).

56.) Show that \( \int \text{sech}(x) \, dx = \tan^{-1}[\text{sech}(x)] + C \).

57.) Evaluate the integral: \[\int \frac{x^3 + x^2 + 1}{x^3 + x} \, dx.\] Possible answer: \( x + \ln(x) - \tan^{-1}(x) \)

58.) Use \( \sum_{k=1}^{n} \sin[k \pi x] = \csc[\frac{\pi}{2}] \sin[\frac{m \pi}{2}] \sin[(n + 1) \frac{\pi}{2}] \) to evaluate the right side Riemann sum for the integral of \( \sin[x] \) from 0 to \( \pi \) when the interval is divided into \( n \) equal width bins. \( \csc[\frac{\pi}{2n}] \sin[\frac{\pi}{2}] \sin[(n + 1) \frac{\pi}{2n}] \) Use L'Hospital to evaluate the limit of \( \frac{\pi}{n} \csc[\frac{\pi}{2n}] \) as \( n \to \infty \). \textbf{NOT CHECKED}

59.) Evaluate \( \int \frac{(A-x) \, dx}{B^2 + (A-x)^2} \) \( \frac{1}{2} \right) \) and \( \int \frac{dx}{B^2 + (A-x)^2} \) \( \frac{1}{2} \) \( x \tan^{-1}(x) - \frac{1}{2} \ln|1-x^2| \)

60.) Evaluate \( \int ???????? \) \[x \tan^{-1}(x) - \frac{1}{2} \ln|1-x^2|\]
61.) Show \( \int \frac{x}{(x^2+1)^n} \, dx = \frac{-1}{2(n-1)} (x^2 + 1)^{-n+1} \) and \( \int \frac{dx}{(x^2+1)^n} = \int (\cos \theta)^{2n-2} \, d\theta \) where \( \theta = \tan^{-1}(x) \).

62.) Evaluate \( \int \frac{x^6}{(x^2+1)} \, dx \). It can be considered as a degenerate case of partial fractions.

Divide \( x^2 + 1 \) into the numerator to yield a polynomial plus remainder. \( \int \text{Polynomial} + \frac{Ax+B}{(x^2+1)} \, dx \)

\[
\left[ \frac{1}{x} \ln |x| - \frac{C}{x} + \frac{C}{x^2} \right]
\]

63.) Given \( 0 < x < \frac{1}{2} \pi \), compute the integral \( \int_0^{\sin(x)} \frac{dt}{\sqrt{1-t^2}} \) using the change of variable \( t = \sin \theta \).

Repeat the evaluation choosing \( t = \cos \theta \).

64.) Consider the integral \( \int \frac{t \, dt}{\sqrt{t^2+1}} \). Begin by integrating by parts. Next note that \( \frac{t^2+1}{\sqrt{t^2+1}} = \sqrt{t^2+1} \). Stare at the last identity for a while. \( = \frac{1}{2} t \sqrt{1 + t^2} - \frac{\text{ArcSinh}[t]}{2} \). Find an explicit representation of \( \text{ArcSinh} \) in terms of logarithms. \( \Rightarrow \) Solve \( \frac{e^t - e^{-t}}{2} = u \) to find \( t(u) \).

65.) It is given that \( \int_0^\infty \frac{\sin[x]}{x} \, dx = \frac{\pi}{2} \). Show that \( \int_0^\infty \frac{(\sin[x])^2}{x^2} \, dx = \frac{\pi}{2} \) by considering \( I(\varepsilon) = \int_0^\infty \left( \frac{\sin[\varepsilon x]}{x} \right)^2 \, dx \).

66.) Compute \( \int \frac{dx}{x^2 \sqrt{x^2-4}} \). Which trig identity is suggested by the argument of the square root? How do you modify the expression to replace 4 by 1?

67.) Compute \( \int x^2 \sqrt{1 + 4x^2} \, dx \). Try a hyperbolic function substitution. Use various identities to replace the squares of hyperbolic function by hyperbolics of twice the argument. Use identities such as \( \cosh^2(x) = 1 + 2 \sinh^2(x) \). Compare your result to the answer found using Wolfram Alpha. It can be presented in many forms! \( \frac{1}{64} \left[ (2x) \sqrt{1 + 4x^2} \left( 1 + 8x^2 \right) - \sinh^{-1}(2x) \right] \)
References:


3. The Wolfram web site: mathworld.wolfram.com/